

REFINED TROPICAL CURVE COUNTS AND CANONICAL BASES FOR QUANTUM CLUSTER ALGEBRAS

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ABSTRACT. We express the quantum and classical versions of the Gross-Hacking-Keel-Kontsevich theta bases for cluster algebras in terms of certain descendant tropical Gromov-Witten invariants (with Block-Göttsche style weighting for the quantum cases). We give a similar tropical description for the scattering diagrams and obtain new invariance results for the relevant quantum tropical counts. As an application, we prove a conjecture of Fock and Goncharov about the behavior of quantum theta functions at roots of unity under the action of the quantum Frobenius map.

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1. INTRODUCTION

In [GHK11, §0.4], Gross, Hacking, and Keel made the remarkable Frobenius structure conjecture, which predicts that the coordinate ring \widehat{B} of the mirror to a log Calabi-Yau variety (Y, D) , along with a canonical additive basis of “theta functions” in \widehat{B} , can be defined in terms of certain descendant log Gromov-Witten invariants of (Y, D) . The cluster varieties of [FG09] give a large and interesting class of examples of such log Calabi-Yau varieties, including double Bruhat cells of semisimple Lie groups [BFZ05], Grassmanians [Sco06] and more general partial flag varieties [GLS08], various moduli space of local systems on punctured surfaces [FG06], and many other important examples. In this paper, we prove a tropical version of the Frobenius structure conjecture for cluster varieties (**Theorem 3.7**). That is, we express the canonical bases of [GHKK14] in terms of certain counts of tropical curves (descendant tropical Gromov-Witten invariants). The upcoming paper [MR] will relate these tropical invariants to some actual descendant log Gromov-Witten invariants of toric varieties, and later work will combine these results and others to prove the Frobenius structure conjecture for cluster varieties.

More generally, Theorem 3.7 expresses the quantum analogs of the theta bases in terms of certain Block-Göttsche [BG14] type quantum-weighted counts of tropical curves. See §1.2 below for ideas on possible enumerative interpretations of these refined counts. In §4, we use Theorem 3.7 to prove a

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conjecture of Fock and Goncharov [FG09, Conjecture 4.8.6] about a certain symmetry of the quantum theta functions at roots of unity (**Theorem 4.2**).

1.1. Outline of the paper and the main results. In §2 we cover some general background on scattering diagrams, following [GHKK14, §1]. We start with the data of a finite-rank lattice \mathcal{N} , a submonoid P , and a Lie algebra \mathfrak{g} graded by a subset \mathcal{N}^+ of \mathcal{N} and satisfying a compatibility condition with a skew-symmetric form \mathbf{W} on \mathcal{N} . Let $\mathcal{M} := \mathcal{N}^*$ (the dual lattice), and let $\pi_1 : n \mapsto \mathbf{W}(n, \cdot)$. A scattering diagram \mathfrak{D} consists of a collection of “walls” ($\mathfrak{d} \subset \mathcal{M}_{\mathbb{R}}, g_{\mathfrak{d}}$) where $g_{\mathfrak{d}}$ is an element of a certain completion of $\exp(\mathfrak{g})$, and $\mathcal{M}_{\mathbb{R}} := \mathcal{M} \otimes \mathbb{R}$. For us, \mathfrak{g} will be the quantum torus Lie algebra, and our “initial” scattering diagrams will be of the form:

$$\mathfrak{D}_{\text{in}} := \{(f_i^\perp, \Psi_{q^{1/d_i}}(z^{f_i})) | i \in I \setminus F\}.$$

Here, $\{f_i\}_{i \in I}$ is a basis for \mathcal{N} indexed by a set I , F is a subset of I , the d_i ’s are certain positive rational numbers, and $\Psi_q(x)$ is essentially the exponential of the quantum dilogarithm (cf. Equation 13). One of the most fundamental theorems about scattering diagrams (cf. Theorem 2.4) says that we can associate to \mathfrak{D}_{in} a “consistent” scattering diagram \mathfrak{D} .

Using \mathfrak{D} and the broken lines construction described in §2.3, we can construct a “mirror” algebra \widehat{B} generated by “theta functions” $\{\vartheta_p\}_{p \in \mathcal{N}}$, as well as a subalgebra \widehat{B}_P generated by $\{\vartheta_p\}_{p \in P}$. Each generic $Q \in \mathcal{M}_{\mathbb{R}}$ determines an embedding of \widehat{B} into a ring $\mathbb{k}_q((\mathcal{N}))$ of q -commuting formal Laurent series. We write $\vartheta_{p,Q}$ for the image of ϑ_p —thus, $\vartheta_{p,Q}$ is a certain formal linear combination of q -commuting monomials z^n , $n \in \mathcal{N}$. Each element of \widehat{B} can be expressed uniquely as a (possibly formal) sum of these theta functions. Thus, \widehat{B} and the theta functions can be described either by giving each $\vartheta_{p,Q}$ for some Q , or by giving the coefficient $\alpha(p_1, \dots, p_s; n)$ of ϑ_n in $\vartheta_{p_1} \cdots \vartheta_{p_s}$ for each $p_1, \dots, p_s, n \in \mathcal{N}$. In fact, we show in **Theorem 2.11** that it suffices to only specify each $\alpha(p_1, \dots, p_s; 0)$. One may view this as saying that \widehat{B} has the structure of an “infinite dimensional Frobenius algebra,” where the trace is the function mapping an element to its constant term.

The following is our main theorem (Theorem 3.7 in the main text). It holds **for both the classical and quantum versions of the theta functions** provided that we use the appropriate classical or quantum versions of $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ and $R_{\mathbf{w}}$ (defined below).

Theorem 1.1. *For $p_1, \dots, p_s \in \mathcal{N} \setminus \ker(\pi_1)$, $s \geq 1$, $n \in \mathcal{N}$, and $Q \in \overline{\mathcal{M}}_{\mathbb{R}} := \pi_1(\mathcal{N}_{\mathbb{R}})$ generic and sufficiently far from the origin, the coefficient of z^n in the product $\vartheta_{p_1, Q} \cdots \vartheta_{p_s, Q}$ is*

$$(1) \quad \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)} N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|}.$$

If Q is sufficiently far from the origin but close to the ray ρ_n generated by $\pi_1(n)$, or if $\mathbf{W}(n, \mathcal{N}^+) = 0$ and Q is any generic point (not necessarily far from the origin), then (1) gives the structure constant $\alpha(p_1, \dots, p_s; n)$.

The sum in Equation 1 is over weight-vectors $\mathbf{w} := (w_{ij})$, $i \in I \setminus F$, $1 \leq j \leq l_i$, such that $\sum_{i,j} w_{ij} f_i + \sum_i p_i = n$. $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ is a weighted¹ count of tropical curves or tropical disks in $\overline{\mathcal{M}}_{\mathbb{R}}$ which, for each i, j , go to infinity parallel to $\pi_1(f_i)$ with multiplicity $w_{ij} |\pi_1(f_i)|$ (where $|u|$ denotes the

¹See Def. 3.3 for the appropriate weighting of the tropical curves. The classical version generalizes that of Mikhalkin [Mik05] from dimension 2 ([MR] will show that this weighting gives the correct log GW numbers), and the quantum version is the corresponding generalization of the Block-Göttsche weights. The quantum-weighting for ψ -classes is new and perhaps surprising since it depends on a choice of ordering of certain marked points.

index of u) in some generically specified hyperplane \mathfrak{d}_{ij} parallel to f_i^\perp , and also go to infinity parallel to $\pi_1(p_i)$ with weight equal to $|\pi_1(p_i)|$ for each $i = 1, \dots, s$. Furthermore, these tropical curves are required to contain an s -valent vertex at $Q \in \overline{\mathcal{M}}_{\mathbb{R}}$. We refer to §3.3 for more details, including the definitions of $R_{\mathbf{w}}$ and $|\text{Aut}(\mathbf{w})|$. We note that a relationship between broken lines and tropical disks in the classical limit can also be found in [CPS, Prop. 5.15] for scattering diagrams in more general affine manifolds.

Geometrically, a tropical curve going to infinity parallel to u with weight w means that the corresponding log curves intersect the divisor D_u at a marked point with intersection multiplicity w . Going to infinity in \mathfrak{d}_{ij} corresponds to saying that this intersection with $D_{\pi_1(f_i)}$ is contained in $\{z^{f_i} + 1 = a_{ij}\}$ for some generically specified a_{ij} . The s -valence of the vertex at Q corresponds to requiring that the log curve has a marked point satisfying a generic point condition and a ψ^{s-2} -condition. This ψ^{s-2} -condition can be viewed as fixing the image of the forgetful map to $\overline{\mathcal{M}}_{0,s+1}$ which remembers only the underlying curve and the marked points corresponding to Q and to the p_i 's. The fact that the classical version of $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ really corresponds to such a count of log curves will be proven in [MR].

We note that these log curves appear to be in the wrong space. Indeed, they live in (a toric compactification of) $\overline{\mathcal{M}} \otimes \mathbb{C}^*$, but the Frobenius structure conjecture says they should live in the cluster variety constructed by gluing together copies of $\mathcal{N} \otimes \mathbb{C}^*$ and compactifying, (or, from the viewpoint of [GHK13], the variety constructed by blowing up a toric compactification of $\mathcal{N} \otimes \mathbb{C}^*$ at the intersection of its boundary with $\{z^{\pi_1(f_i)} + 1 = 0 \mid i \in I \setminus F\}$). Pulling back from $\overline{\mathcal{M}}$ to \mathcal{N} is easy using Remark 2.10, which roughly says that we can take π_1^{-1} of everything. Passing from invariants of $\mathcal{N} \otimes \mathbb{C}^*$ to invariants of the cluster variety is trickier and will be done in a later paper using the degeneration techniques of [GPS10, §5.3]. The $\frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|}$ -factors will show up in the degeneration formula. This will complete the proof of the Frobenius structure conjecture for cluster varieties.

In the process of proving Theorem 1.1, we obtain a new invariance result for the relevant quantum descendant tropical curve counts (**Theorem 3.12**). Another invariance result for Block-Göttsche counts was previously obtained by [IM13] in dimension 2 and arbitrary genus. Our method is quite different, based entirely on properties of scattering diagrams.

In **Theorem 3.13**, we describe the scattering diagram \mathfrak{D} itself in terms of counts of tropical disks and tropical curves. This generalizes [GPS10, Theorem 2.8] to the higher-dimensional quantum setting (the two-dimensional quantum version is [FS15, Corollary 4.9]). In future work we plan to give an interpretation of these tropical curve counts in terms of log Gromov-Witten invariants, thus generalizing the main results of [GPS10] and proving a DT/GW-type correspondence between these invariants and those of [Rei10] and [Bri16].

In §4, we prove Conjectures 4.2(ii) and 4.8.6 (with some conditions) of Fock and Goncharov [FG09]. The first of these (our **Theorem 4.1**) simply states that for the classical theta functions with $n \in \mathcal{N}$ and p a prime, $\vartheta_{pn} \equiv \vartheta_n^p \pmod{p}$. This turns out to be straightforward using the definition of the theta functions in terms of broken lines. The latter result (our **Theorem 4.2**) is a sort of quantum analog. It involves Fock and Goncharov's quantum Frobenius map and is a symmetry relating powers of quantum theta functions at roots of unity to the classical theta functions. Our proof of this relies heavily on the quantum and classical versions of our Theorem 1.1. We note that the case of quantum cluster varieties from surfaces is [AK15, Theorem 1.2.6], assuming that their canonical bases turn out to equal the theta bases.

In §5 we give background on the cluster varieties \mathcal{A} and \mathcal{X} of [FG09], along with their quantizations \mathcal{A}_q and \mathcal{X}_q . These spaces are constructed from a base seed S , which consists of the data of a finite-rank lattice N , a basis $E = \{e_i\}_{i \in I}$ indexed by a set I , a subset $F \subset I$, a \mathbb{Q} -valued skew-form $\{\cdot, \cdot\}$ on N , and positive rational numbers $\{d_i\}_{i \in I}$ such that each $\epsilon_{ij} := d_j \{e_i, e_j\}$ is an integer. For an appropriate choice of scattering diagram $\mathfrak{D}_{\mathcal{V}_q}$ (\mathcal{V} standing in for either \mathcal{A} or \mathcal{X}), the associated algebra of theta functions $\widehat{B}_{\mathcal{V}_q}$ is closely related to (possibly equal to) \mathcal{V}_q . We describe this choice of scattering diagram in §5.2, and in §5.3 we sketch some aspects of how $\widehat{B}_{\mathcal{V}}$ relates to \mathcal{V} , referring to [GHKK14] for more details on this relationship. We include some ideas about this relationship for the quantum analogs, but we do not attempt to prove any precise details about this. We briefly describe the appropriate choices of scattering diagrams here:

\mathcal{A}_q : Recall that quantizing the \mathcal{A} -space (i.e., quantizing the cluster algebra) requires the existence of a form Λ on $M := N^*$ which is “compatible” with the form $B(e_i, e_j) := \epsilon_{ij}$, cf. §5.1 or [BZ05] for more details. To construct the algebra of theta functions $\widehat{B}_{\mathcal{A}_q}$ associated with \mathcal{A}_q , one uses the scattering diagram $\mathfrak{D}_{\mathcal{A}_q, \text{in}}$ defined as follows: We take $\mathcal{N} := M$ and $\mathbf{W} := \Lambda$. I and F are as in the seed data, and our basis $\{f_i | i \in I\}$ for M is chosen so that $f_i := B_1(e_i) := B(e_i, \cdot)$ (i.e., the i -th row of B) for $i \in I \setminus F$. We take d_i as in the seed data for $i \in I \setminus F$. There is some flexibility in the f_i ’s and d_i ’s for $i \in F$. Our initial scattering diagram is then:

$$\mathfrak{D}_{\mathcal{A}_q, \text{in}} := \left\{ (B_1(e_i)^\perp, [\Psi_{q^{1/d_i}}(z^{B_1(e_i)})]^{-1}) | i \in I \setminus F \right\}.$$

\mathcal{X}_q : Similarly, when constructing the analogous algebra $\widehat{B}_{\mathcal{X}_q}$ for the quantized \mathcal{X} -space, we take $\mathcal{N} := N$, $\mathbf{W} := \{\cdot, \cdot\}$, and $I, F, \{d_i | i \in I\}$ as in the seed data. The basis $\{f_i | i \in I\}$ for \mathcal{N} is also the one from the seed data—i.e., $f_i := e_i$. The initial scattering diagram is then:

$$\mathfrak{D}_{\mathcal{X}_q, \text{in}} := \left\{ (e_i^\perp, \Psi_{q^{1/d_i}}(z^{e_i})) | i \in I \setminus F \right\}.$$

$\text{mid}(\mathcal{A})$: [GHKK14] describes a subset $\Theta_{\mathcal{A}} \subset M$ such that $\{\vartheta_p\}_{p \in \Theta_{\mathcal{A}}}$ generates a subalgebra $\text{mid}(\mathcal{A})$ of the (non-quantized) upper cluster algebra $\text{up}(\mathcal{A})$. However, if S does not satisfy a certain convexity condition (which holds whenever a compatible Λ as above exists), then the methods of [GHKK14] do not allow one to construct $\text{mid}(\mathcal{A})$ and these theta functions directly. Rather, one works first with the corresponding cluster algebra with principal coefficients and then sets these coefficients equal to 1. On the other hand, we see that the tropical curves of Theorem 1.1 can be defined more directly. **Theorem 5.4** defines the theta functions in $\text{mid}(\mathcal{A})$ directly in terms of tropical curves/disks in $M_{\mathbb{R}}$ without the need for $\mathcal{A}^{\text{prin}}$. In particular, this expresses any product of cluster monomials in terms of counts of tropical curves and disks!

1.2. Refined invariants. There are motivic interpretations of various Block-Göttsche counts in dimension 2, c.f., [GS14] and [NPS16]. A motivic interpretation in our setup (i.e., for certain theta functions and scattering diagrams) is given by the approaches of [Rei10] and [Bri16]. See [FS15] for a direct connection between this motivic approach and Block-Göttsche counts in certain 2-dimensional cases. We expect that a more direct enumerative geometric description of these refined counts should also exist. In particular, some of the quantum-weighted tropical curve counts relevant to 2-dimensional scattering diagrams (as in [FS15, Corollary 4.9], which is the 2-dimensional case of our Theorem 3.13) have been related to certain real Gromov-Witten invariants in [Mik16]. We hope to eventually extend Mikhalkin’s approach and combine it with our main result to prove a quantum version of the Frobenius structure conjecture in terms of a new type of real/open descendant Gromov-Witten invariant.

1.3. Generalizations. This paper focuses on scattering diagrams over the quantum torus Lie algebra of the form given in (16), along with their classical limits. However, generalizations of our results to other scattering diagrams and even other Lie algebras \mathfrak{g} should be straightforward using our methods. The key ingredients needed for generalization are in Remarks 2.6 and 3.4 (the latter explains how to generalize the tropical multiplicities). The $R_{w,d_i;q}$'s should be replaced with the corresponding coefficients in the analog of (21). In particular, we expect that versions of Theorems 3.7 and 3.13 should hold for the Hall algebra scattering diagrams (assuming genteel potentials) of [Bri16] and the corresponding broken lines and theta functions of [Che16].

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1.5. Notation and preliminaries. We use the following notation throughout the paper. \mathbb{k} denotes an algebraically closed field of characteristic 0 (although this can often be weakened). \mathcal{N} denotes a rank r lattice, $\mathcal{M} := \text{Hom}(\mathcal{N}, \mathbb{Z})$ the dual lattice, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing between \mathcal{N} and \mathcal{M} . We write $\mathcal{N}_{\mathbb{R}}$ and $\mathcal{N}_{\mathbb{Q}}$ for $\mathcal{N} \otimes \mathbb{R}$ and $\mathcal{N} \otimes \mathbb{Q}$, respectively. An element $v \in \mathcal{N}$ is called **primitive** if it is not a positive multiple of any other element of \mathcal{N} . If $v \in \mathcal{N}$ is $k > 0$ times a primitive vector in \mathcal{N} , we call k the **index** of v and write $|v| := k$.

1.5.1. Module completions and structure constants. For any set S and commutative ring R , we denote by $R\langle S \rangle$ the free R -module with basis $\{\vartheta_p\}_{p \in S}$. That is,

$$R\langle S \rangle := \bigoplus_{p \in S} R \cdot \vartheta_p.$$

Suppose S is a finitely generated monoid containing a set \mathcal{I} which is closed under addition by elements of S and does not contain 0. Denote

$$(2) \quad k\mathcal{I} := \{p \in S \mid p = kp_0 + p' \text{ for some } p_0 \in \mathcal{I}, p' \in S\}.$$

Let S^\times denote the invertible elements of S . S admits a filtration $S^\times =: F_0 \subseteq F_1 \subseteq \dots \subseteq S$, $S = \bigcup F_i$, given by $F_k := S \setminus k\mathcal{I}$ for $k \geq 1$. We denote by $R\langle\langle S \rangle\rangle_{\mathcal{I}}$ the completion of $R\langle S \rangle$ with respect to this filtration—that is, $R\langle\langle S \rangle\rangle_{\mathcal{I}}$ is the R -module consisting of infinite sums of elements of $R\langle S \rangle$ such that for each $k \geq 0$, only finitely many terms of the sum are contained in F_k .

One can define a multiplicative structure on $R\langle\langle S \rangle\rangle_{\mathcal{I}}$ by identifying the **structure constants** $\alpha(p_1, \dots, p_s; p) \in R$:

$$(3) \quad \vartheta_{p_1} \cdots \vartheta_{p_s} := \sum_{p \in S} \alpha(p_1, \dots, p_s; p) \vartheta_p.$$

Given an algebra structure on $R\langle\langle S \rangle\rangle$, we let

$$\overline{R\langle\langle S \rangle\rangle}$$

denote the resulting subalgebra generated by $\{\vartheta_p\}_{p \in S}$. Note that $R\langle S \rangle \subseteq \overline{R\langle\langle S \rangle\rangle}$ as modules, with equality if and only if each $\vartheta_{p_1} \vartheta_{p_2}$ is a *finite* linear combination of ϑ_p 's.

1.5.2. *Quantum tori.* Let σ_P be a convex rational polyhedral cone in $\mathcal{N}_{\mathbb{R}}$, not necessarily strictly convex, but not all of $\mathcal{N}_{\mathbb{R}}$. Let P be the submonoid $\sigma_P \cap \mathcal{N} \subset \mathcal{N}$, and P^\times the group of invertible elements in P . Let $\mathcal{I} := P \setminus P^\times$, and define $k\mathcal{I}$ as in (2). Suppose \mathcal{N} is equipped with a \mathbb{Q} -valued skew-symmetric bilinear form \mathbf{W} . Let D be the smallest positive rational number such that $D\mathbf{W}$ is \mathbb{Z} -valued on \mathcal{N} . Let R be a commutative ring containing an indeterminant $q^{1/D}$ and its inverse. We consider the non-commutative polynomial ring $R[\mathcal{N}]$ generated over R by z^n , $n \in \mathcal{N}$, with multiplication defined by

$$(4) \quad z^{n_1} z^{n_2} = q^{\mathbf{W}(n_1, n_2)} z^{n_1 + n_2}.$$

In particular, we consider the ring $\mathbb{k}_q := \mathbb{k}[q^{\pm 1/D}]$ and the non-commutative torus

$$\mathcal{T}_{\mathcal{M}, q^{1/D}, \mathbf{W}} := \mathbb{k}_q[\mathcal{N}].$$

Note that taking the $q^{1/D} \rightarrow 1$ limit recovers the classical torus algebra.

We similarly define $R[P]$ using only z^p with $p \in P$. Abusing notation, we let $k\mathcal{I}$ denote the monomial ideal of $R[P]$ corresponding to the monoid ideal $k\mathcal{I}$. We denote by $R[[P]]_{\mathcal{I}}$ the completion of $R[P]$ with respect to \mathcal{I} (so $R\langle\langle P \rangle\rangle_{\mathcal{I}} = R[[P]]_{\mathcal{I}}$ as modules). We write $R((\mathcal{N}))_{\mathcal{I}}$ for the formal Laurent series ring spanned by $\{z^n R[[P]]_{\mathcal{I}} | n \in \mathcal{N}\}$. We may leave off the subscript if \mathcal{I} is clear from context.

We will also make heavy use of the notation

$$(5) \quad [w]_q := \frac{q^w - q^{-w}}{q - q^{-1}}.$$

Note that $\lim_{q \rightarrow 1} [w]_q = w$.

2. THETA BASES FROM BROKEN LINES

Here we review [GHKK14]'s construction of theta functions in terms of broken lines, including adding details to the quantum version of their construction.

2.1. Scattering diagrams. We first recall some basic definitions and theorems regarding scattering diagrams, following [GHKK14, §1]. Notation is as introduced in §1.5 above. Let $\{f_i\}_{i \in I}$ be a basis for \mathcal{N} with index set I . For some subset $F \subset I$, let

$$\mathcal{N}^+ := \left\{ \sum_{i \in I \setminus F} a_i f_i \mid a_i \in \mathbb{Z}_{\geq 0} \right\} \setminus \{0\}.$$

Let $\mathfrak{g} := \bigoplus_{n \in \mathcal{N}^+} \mathfrak{g}_n$ be a Lie algebra over \mathbb{k} graded by \mathcal{N}^+ , meaning that $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] \subseteq \mathfrak{g}_{n_1 + n_2}$. Define $d : \mathcal{N} \rightarrow \mathbb{Z}$ by

$$(6) \quad d\left(\sum_{i \in I} a_i f_i\right) := \sum_{i \in I} a_i,$$

and define

$$\mathfrak{g}^{>k} := \bigoplus_{d(n) > k} \mathfrak{g}_n.$$

Then $\mathfrak{g}^{\leq k} := \mathfrak{g} / \mathfrak{g}^{>k}$ is a nilpotent Lie algebra. Let $G^{\leq k} := \exp(\mathfrak{g}^{\leq k})$ be the corresponding nilpotent Lie group, and $G := \varprojlim_k G^{\leq k}$. In other words, $G = \exp(\widehat{\mathfrak{g}})$, where $\widehat{\mathfrak{g}} := \varprojlim \mathfrak{g}^{\leq k}$.

For each $n \in \mathcal{N}^+$, we have a Lie subalgebra $\mathfrak{g}_n^{\parallel} := \prod_{k \in \mathbb{Z}_{>0}} \mathfrak{g}_{kn} \subset \widehat{\mathfrak{g}}$, and a Lie subgroup $G_n^{\parallel} := \exp(\mathfrak{g}_n^{\parallel}) \subset G$. Assume that each $\mathfrak{g}_n^{\parallel}$ (and thus each G_n^{\parallel}) is Abelian.

Definition 2.1. For the above data, a **wall** in $\mathcal{M}_{\mathbb{R}}$ is a pair $(\mathfrak{d}, g_{\mathfrak{d}})$ such that:

- $g_{\mathfrak{d}} \in G_{n_{\mathfrak{d}}}^{\parallel}$ for some primitive $n_{\mathfrak{d}} \in \mathcal{N}$.
- For some $m_{\mathfrak{d}} \in \mathcal{M}_{\mathbb{R}}$, $\mathfrak{d} \subseteq m_{\mathfrak{d}} + n_{\mathfrak{d}}^{\perp} \subset \mathcal{M}_{\mathbb{R}}$ is an $(r-1)$ -dimensional convex (but not necessarily strictly convex) rational polyhedral affine cone, called the **support** of the wall.

We note that the data of the wall uniquely determines $n_{\mathfrak{d}}$, but does not uniquely determine $m_{\mathfrak{d}}$. A **scattering diagram** \mathfrak{D} is a set of walls such that for each $k > 0$, there are only finitely many $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ with $g_{\mathfrak{d}}$ not projecting to the identity in $G^{\leq k}$.

We will sometimes denote a wall $(\mathfrak{d}, g_{\mathfrak{d}})$ by just \mathfrak{d} . Denote $\text{Supp}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathfrak{d}$, and

$$\text{Joints}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \subset \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \in \mathfrak{D} \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = r-2}} \mathfrak{d}_1 \cap \mathfrak{d}_2.$$

Note that for each $k > 0$, a scattering diagram \mathfrak{D} for \mathfrak{g} induces a finite scattering diagram \mathfrak{D}^k for $\mathfrak{g}^{\leq k}$ with walls corresponding to the $g_{\mathfrak{d}}$ which are nontrivial in $G^{\leq k}$.

Consider a smooth immersion $\gamma : [0, 1] \rightarrow \mathcal{M}_{\mathbb{R}} \setminus \text{Joints}(\mathfrak{D})$ with endpoints not in $\text{Supp}(\mathfrak{D})$ which is transverse to each wall of \mathfrak{D} it crosses. Let $(\mathfrak{d}_i, g_{\mathfrak{d}_i})$, $i = 1, \dots, s$, denote the walls of \mathfrak{D}^k crossed by γ , and say they are crossed at times $0 < t_1 \leq \dots \leq t_s < 1$, respectively (if $t_i = t_{i+1}$, then the assumption that each $\mathfrak{g}_n^{\parallel}$ is Abelian implies that the ordering of these two walls does not matter). Define

$$(7) \quad \theta_{\mathfrak{d}_i} := g_{\mathfrak{d}_i}^{\text{sgn}\langle n_{\mathfrak{d}}, -\gamma'(t_i) \rangle} \in G^{\leq k}.$$

Let $\theta_{\gamma, \mathfrak{D}}^k := \theta_{\mathfrak{d}_s} \cdots \theta_{\mathfrak{d}_1} \in G^{\leq k}$, and define the **path-ordered product**:

$$\theta_{\gamma, \mathfrak{D}} := \varprojlim_k \theta_{\gamma, \mathfrak{D}}^k \in G.$$

Definition 2.2. Two scattering diagrams \mathfrak{D} and \mathfrak{D}' are **equivalent** if $\theta_{\gamma, \mathfrak{D}} = \theta_{\gamma, \mathfrak{D}'}$ for each smooth immersion γ as above. \mathfrak{D} is **consistent** if each $\theta_{\gamma, \mathfrak{D}}$ depends only on the endpoints of γ .

Suppose \mathcal{N} is equipped with a \mathbb{Q} -valued skew-symmetric form $\mathbf{W}(\cdot, \cdot)$. \mathfrak{g} is called **skew-symmetric** (with respect to \mathbf{W}) if $\mathbf{W}(n_1, n_2) = 0$ implies $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] = 0$ for each $n_1, n_2 \in \mathcal{N}^+$. Define

$$\pi_1(n) := \mathbf{W}(n, \cdot) \in n^{\perp} \subset \mathcal{M}_{\mathbb{Q}}.$$

Definition 2.3. With notation as in Definition 2.1, a wall $(\mathfrak{d}, g_{\mathfrak{d}})$ is called **incoming** if $m_{\mathfrak{d}} + t\pi_1(n_{\mathfrak{d}}) \in \mathfrak{d}$ for all sufficiently large $t \in \mathbb{R}_{>0}$, and **outgoing** otherwise. $-\pi_1(n_{\mathfrak{d}})$ is called the **direction** of the wall.

The following fundamental theorem on scattering diagrams is due to [KS06] in dimension 2 and [GS11] and [KS13] in higher dimensions, and appears as Theorem 1.7 in [GHKK14]:

Theorem 2.4. *Let \mathfrak{g} be a skew-symmetric \mathcal{N}^+ -graded Lie algebra, and let \mathfrak{D}_{in} be a finite scattering diagram whose only walls have full affine hyperplanes as their supports. Then there is a scattering diagram $\mathfrak{D} =: \text{Scat}(\mathfrak{D}_{\text{in}})$, unique up to equivalence, such that \mathfrak{D} is consistent, $\mathfrak{D} \supset \mathfrak{D}_{\text{in}}$, and $\mathfrak{D} \setminus \mathfrak{D}_{\text{in}}$ consists only of outgoing walls.*

2.2. Scattering diagrams over the quantum torus. Here we follow [GHKK14, Construction 1.31]. Let $\{d_i\}_{i \in I}$ be positive rational numbers such that $d_i \pi_1(f_i) \in \mathcal{M}$ (as opposed to just $\mathcal{M}_{\mathbb{Q}}$). Let D be the smallest (or more generally, any) positive rational number such that $D\mathbf{W}$ is \mathbb{Z} -valued on \mathcal{N} . Recall the notation $\mathbb{k}_q := \mathbb{k}[q^{\pm 1/D}]$, $R[\mathcal{N}]$, and $[w]_q$ from §1.5.2. Also, choose $\sigma_P \supset \mathcal{N}^+$, $P := \sigma_P \cap \mathcal{N}$, and $\mathcal{I} := P \setminus P^\times$ as in 1.5.2. For example, one may take $P := \{\sum_{i \in I} a_i f_i \mid a_i \in \mathbb{Z}, \text{ and if } i \notin F, \text{ then } a_i \geq 0\}$.

Observe that $\mathbb{k}(q^{\pm 1/D})[\mathcal{N}]$ forms a Lie algebra \mathfrak{g}_0 under the negation of the standard commutator:

$$[x, y] := yx - xy.$$

Let $\mathbb{k}_q^\circ \subset \mathbb{k}(q^{1/D})$ denote the localization $S^{-1}\mathbb{k}_q$, where $S \subset \mathbb{k}_q$ is the complement of the union of the prime ideals of the form $\langle q^{1/D} - \zeta \rangle$, ζ a root of unity.² Then define $\mathfrak{g} \subset \mathbb{k}(q^{1/D})[\mathcal{N}]$ to be the free \mathbb{k}_q° -submodule with basis $\{\hat{z}^n \mid n \in \mathcal{N}^+\}$, where

$$\hat{z}^n := \frac{z^n}{q - q^{-1}}.$$

This is a Lie subalgebra of \mathfrak{g}_0 over \mathbb{k}_q° with bracket:

$$[\hat{z}^{n_1}, \hat{z}^{n_2}] = [\mathbf{W}(n_2, n_1)]_q \hat{z}^{n_1+n_2}.$$

One easily sees that \mathfrak{g} is a skew-symmetric \mathcal{N}^+ -graded Lie algebra. The (semi-)classical limit is obtained by sending $\hat{z}^n \mapsto z^n$ and $q^{1/D} \mapsto 1$. See [GHKK14, Construction 1.21] for more on this classical version.

Exponentiating the adjoint action $\text{ad} : g \mapsto [g, \cdot]$ of $\hat{\mathfrak{g}}$ induces actions Ad of G on $\mathbb{k}_q[[P]]$, $\mathbb{k}_q((\mathcal{N}))$, and $\mathbb{k}_q[P]/k\mathcal{I}$. These actions are by conjugation:

$$\text{Ad}(\exp(g)) : f \mapsto \exp(-g)f\exp(g).$$

For any commutative \mathbb{k}_q -algebra T , we can replace the Lie algebra \mathfrak{g} above with $\mathfrak{g} \otimes_{\mathbb{k}_q} T$, taking the Lie bracket defined by $[g_1 \otimes t_1, g_2 \otimes t_2] := [g_1, g_2]_{\mathfrak{g}} \otimes (t_1 t_2)$. We denote the corresponding limit of Lie groups as in §2.1 by $G \hat{\otimes} T$ (the hat is meant to indicate that the tensor product should take place before the completion). The Ad action of G on $\mathbb{k}_q((\mathcal{N}))$ becomes an action on $T((\mathcal{N}))$.

The following lemma is a higher-dimensional quantum analog of Lemma 1.9 in [GPS10]. The two-dimensional quantum version is Lemma 4.3 in [FS15].

Lemma 2.5. *Let T be a commutative \mathbb{k}_q -algebra with $t_1, t_2 \in T$, $t_1^2 = t_2^2 = 0$. Let $n_1, n_2 \in \mathcal{N}^+$, so $t_i \hat{z}^{n_i} \in \mathfrak{g} \otimes T$. Then in $G \hat{\otimes} T$, we have*

$$(8) \quad (1 - t_2 \hat{z}^{n_2})(1 + t_1 \hat{z}^{n_1})(1 + t_2 \hat{z}^{n_2})(1 - t_1 \hat{z}^{n_1}) = 1 + t_1 t_2 [\mathbf{W}(n_1, n_2)]_q \hat{z}^{n_1+n_2}.$$

If $\mathfrak{D}_{\text{in}} := \{(\mathfrak{d}_1, g_{\mathfrak{d}_1} := 1 + t_1 \hat{z}^{n_1}), (\mathfrak{d}_2, g_{\mathfrak{d}_2} := 1 + t_2 \hat{z}^{n_2})\}$ is a scattering diagram for $G \hat{\otimes} T$, then either $\mathbf{W}(n_1, n_2) = 0$ and \mathfrak{D}_{in} is consistent, or

$$\mathfrak{D}' := \mathfrak{D}_{\text{in}} \cup \{(\mathfrak{d} := (\mathfrak{d}_1 \cap \mathfrak{d}_2) + \mathbb{R}_{\leq 0} \pi_1(n_1 + n_2), g_{\mathfrak{d}} := 1 + t_1 t_2 [\mathbf{W}(n_1, n_2)]_q \hat{z}^{n_1+n_2})\}$$

is consistent in $G \hat{\otimes} T$.

Proof. The first part is a straightforward computation (alternatively, see Remark 2.6). The second part then follows easily by working through what the definition of consistency means in this situation. \square

²An earlier version of this paper only included $\zeta = \pm 1$, but all roots of unity are needed for the functions $\Psi_{q^{1/d_i}}(z^{f_i})$ in (16) actually live in G .

Remark 2.6. We note that Equation 8 can also be interpreted as

$$(9) \quad [\exp(t_1 x), \exp(t_2 y)]_{G \widehat{\otimes} T} = \exp([t_1 x, t_2 y]_{\mathfrak{g} \widehat{\otimes} T})$$

with $[a, b]_{G \widehat{\otimes} T} := bab^{-1}a^{-1}$, $x := \hat{z}^{n_1}$, and $y := -\hat{z}^{n_2}$. One can check using the Baker-Campbell-Hausdorff formula that (9) holds for any Lie algebra $\mathfrak{g} \widehat{\otimes} T$ over an algebra T in which $t_1^2 = t_2^2 = 0$, and any $x, y \in \mathfrak{g}$. Lemma 2.5 can thus be generalized to other \mathcal{N}^+ -graded Lie algebras \mathfrak{g} . By combining this with Remark 3.4 and a generalization of Lemma 2.8, it should be straightforward to generalize the main results of this paper to other \mathcal{N}^+ -graded Lie algebras \mathfrak{g} .

All scattering diagrams from now on will be assumed to be over the quantum torus Lie algebra \mathfrak{g} defined above or over $\mathfrak{g} \widehat{\otimes} T$ for some \mathbb{k}_q -algebra T as above.

2.3. Broken lines.

Definition 2.7. Let $p \in \mathcal{N} \setminus \ker(\pi_1)$ and $Q \in \mathcal{M}_{\mathbb{R}} \setminus \mathfrak{D}$. A **broken line** γ with **ends** (p, Q) is the data of a continuous map $\gamma : (-\infty, 0] \rightarrow \mathcal{M}_{\mathbb{R}} \setminus \text{Joints}(\mathfrak{D})$, values $-\infty < t_0 \leq t_1 \leq \dots \leq t_\ell = 0$, and for each $i = 0, \dots, \ell$, an associated monomial $c_i z^{v_i} \in \mathbb{k}_q((\mathcal{N}))$ with $c_i \in \mathbb{k}_q$, $v_i \in \mathcal{N} \setminus \ker(\pi_1)$, such that:

- $\gamma(0) = Q$.
- For $i = 1, \dots, \ell$, $\gamma'(t) = -\pi_1(v_i)$ for all $t \in (t_{i-1}, t_i)$. Similarly, $\gamma'(t) = -\pi_1(v_0)$ for all $t \in (-\infty, t_0)$.
- $c_0 z^{v_0} = z^p$.
- For $i = 0, \dots, \ell - 1$, $\gamma(t_i)$ is contained in a scattering ray $(\mathfrak{d}_i, g_{\mathfrak{d}_i} \in G_{n_{\mathfrak{d}_i}}^{\parallel}) \in \mathfrak{D}$, and $c_{i+1} z^{v_{i+1}}$ is a monomial term in

$$(10) \quad [\text{Ad}(g_{\mathfrak{d}_i})^{\text{sgn}(\mathbf{W}(v_i, n_{\mathfrak{d}_i}))}](c_i z^{v_i}),$$

viewed as a formal Laurent series with coefficients in \mathbb{k}_q . Note that (10) is just the Ad-action of $\theta_{\mathfrak{d}_i}$ as defined as in Equation 7 (with respect to a smoothing of γ). If $i \neq j$, then $(\mathfrak{d}_i, f_{\mathfrak{d}_i})$ and $(\mathfrak{d}_j, f_{\mathfrak{d}_j})$ are distinct walls of \mathfrak{D} , even if $t_i = t_j$. If $t_i = t_j$, we identify the broken line with the one obtained by reordering t_i and t_j .

These broken lines will be used for defining the quantum theta functions. Broken lines for the commutative theta functions of [GHKK14] can be obtained by taking the $q^{1/D} \rightarrow 1$ limits of the attached monomials for these broken lines.

We can similarly define broken lines with G extended to $G \widehat{\otimes} T$, letting the attached monomials live in $T((\mathcal{N}))$. The following lemma expresses (10) in a very simple situation—in fact, this is the only situation we will need. It is easily checked explicitly.

Lemma 2.8. *Let T be a commutative \mathbb{k}_q -algebra with $t \in T$, $t^2 = 0$. Consider $cz^v \in T((\mathcal{N}))$ and $g := 1 + t\hat{z}^n \in G \widehat{\otimes} T$. Then*

$$[\text{Ad}(g)^{\text{sgn}(\mathbf{W}(v, n))}](cz^v) = cz^v + c[\|\mathbf{W}(v, n)\|]_q t z^{v+n}.$$

2.4. Defining the theta functions. Fix a consistent scattering diagram \mathfrak{D} and a point $Q \in \mathcal{M}_{\mathbb{R}} \setminus \text{Supp}(\mathfrak{D})$. For $p \in \ker(\pi_1)$, we define $\vartheta_{p, Q} := z^p \in \mathbb{k}_q((\mathcal{N}))$. For $p \in \mathcal{N} \setminus \ker(\pi_1)$, we define

$$\vartheta_{p, Q} := \sum_{\text{Ends}(\gamma) = (p, Q)} c_\gamma z^{n_\gamma} \in \mathbb{k}_q((\mathcal{N})).$$

Here, the sum is over all broken lines with ends (p, Q) , and $c_\gamma z^{n_\gamma}$ denotes the monomial attached to the final straight segment of γ . The proof of [GHKK14, Prop. 3.14] shows that this sum is well-defined in $\mathbb{k}_q((\mathcal{N}))$. In particular, it shows that there are only finitely many broken lines with ends (p, Q) and fixed final exponent n_γ . We may write $\vartheta_{p,Q}^{\mathfrak{D}}$ if \mathfrak{D} is not clear from context.

Let \widehat{B}_Q denote the subalgebra of $\mathbb{k}_q((\mathcal{N}))$ generated by $\{\vartheta_{p,Q}\}_{p \in \mathcal{N}}$. As an abstract algebra (i.e., not embedded in $\mathbb{k}_q((\mathcal{N}))$), the methods of [CPS] show that this is independent of the choice of Q : for γ a path between Q and Q' , the path-ordered product $\theta_{\gamma, \mathfrak{D}}$ provides a canonical isomorphism between \widehat{B}_Q and $\widehat{B}_{Q'}$, identifying $\vartheta_{p,Q}$ with $\vartheta_{p,Q'}$. We denote this abstract algebra by \widehat{B} , and the theta functions by $\vartheta_p \in \widehat{B}$.

One can show that the theta functions are linearly independent, and any product of theta functions is a \mathbb{k}_q -linear formal sum of theta functions such that we have the \mathbb{k}_q -module inclusions

$$\mathbb{k}_q\langle \mathcal{N} \rangle \subseteq \widehat{B} \subset \mathbb{k}_q\langle\langle \mathcal{N} \rangle\rangle.$$

We can define \widehat{B}_P as the subalgebra of \widehat{B} generated by $\{\vartheta_p\}_{p \in P}$. Alternatively, we can define \widehat{B}_P directly in terms of broken lines with ends (p, Q) , $p \in P$, using $\mathbb{k}_q[[P]]$ in place of $\mathbb{k}_q((\mathcal{N}))$. We have the analogous containments

$$\mathbb{k}_q\langle P \rangle \subseteq \widehat{B}_P \subset \mathbb{k}_q\langle\langle P \rangle\rangle.$$

We similarly define B_k using $\mathbb{k}_q[P]/k\mathcal{I}$. Note that $B_k = \mathbb{k}_q\langle P \setminus k\mathcal{I} \rangle$ as \mathbb{k}_q -modules, and $\widehat{B}_P = \varprojlim_k B_k$.

Consider any saturated sublattice $K \subset \mathcal{N}$ such that $\mathbf{W}(K, \mathcal{N}^+) = 0$ and $\ker(\pi_1) \subseteq K$ (e.g., $K = \ker(\pi_1)$). For any $p \in K$, the corresponding broken lines do not bend, so $\vartheta_{p,Q} = z^p$ for all $Q \in \mathcal{M}_{\mathbb{R}} \setminus \text{Supp}(\mathfrak{D})$. This makes \widehat{B} into a $\mathbb{k}_q[K]$ -algebra, with ϑ_k identified with z^k for each $k \in K$. The multiplication rule for the theta functions in \widehat{B} can now be described as follows: for any $k \in K$, $n \in \mathcal{N}$,

$$\vartheta_k \vartheta_n = \vartheta_{k+n},$$

and, for any $p_1, \dots, p_s \notin \ker(\pi_1)$, $p \in \mathcal{N}$, the structure constant $\alpha(p_1, \dots, p_s; p) \in \mathbb{k}_q$ is

$$(11) \quad \alpha(p_1, \dots, p_s; p) := \sum_{\substack{\gamma_1, \dots, \gamma_s \\ \text{Ends}(\gamma_i) = (p_i, Q_{\pi_1(p)}), i=1, \dots, s \\ v_{\gamma_1} + \dots + v_{\gamma_s} = p}} c_{\gamma_1} \cdots c_{\gamma_s}.$$

Here, γ_i , $i = 1, \dots, s$ are broken lines, $c_{\gamma_i} z^{v_{\gamma_i}} \in \mathbb{k}_q((\mathcal{N}))$ is the monomial attached to the last straight segment of γ_i , and $Q_{\pi_1(p)} \in \mathcal{M}_{\mathbb{R}} \setminus \text{Supp}(\mathfrak{D})$ is any point “sufficiently close” to $\pi_1(p)$ —[GHKK14, Def.-Lemma 6.2] shows that there are only finitely many broken lines contributing to $\alpha(p_1, \dots, p_s; p)$, no matter what choice of $Q_{\pi_1(p)}$ we use. $Q_{\pi_1(p)}$ being sufficiently close to $\pi_1(p)$ means that moving it closer to $\pi_1(p)$ will not change the set of broken lines contributing to this sum. Despite the notation $Q_{\pi_1(p)}$, being sufficiently close to $\pi_1(p)$ will also depend on p_i ’s. If $\mathbf{W}(p, \mathcal{N}^+) = 0$, then $Q_{\pi_1(p)}$ can be any generic point. The analogous formulas hold for \widehat{B}_P and B_k . For B_k , we can take sufficiently close to mean that Q and p are contained in a common cell of \mathfrak{D}^k .

Alternatively, we can take advantage of the $\mathbb{k}_q[K]$ -algebra structure. Let π_K denote the projection

$$\pi_K : \mathcal{N} \rightarrow \overline{\mathcal{N}} := \mathcal{N}/K.$$

Choose a section

$$\tilde{\varphi} : \overline{\mathcal{N}} \rightarrow \mathcal{N}$$

of π_K (viewed as a map of sets) with $\tilde{\varphi}(0) = 0$. For $n \in \overline{\mathcal{N}}$, define

$$\vartheta_n := \vartheta_{\tilde{\varphi}(n)}.$$

Since \widehat{B} is generated as a $\mathbb{k}_q[K]$ -algebra by $\{\vartheta_n | n \in \overline{\mathcal{N}}\}$, when defining the multiplication rule on $\mathbb{k}_q\langle\langle\mathcal{N}\rangle\rangle$ it now suffices to just define the products of *these* theta functions. We have $\vartheta_0 = 1$, and for $p_1, \dots, p_s \in \overline{\mathcal{N}} \setminus \{0\}$, we have the following analog of (11):

$$(12) \quad \vartheta_{p_1} \cdots \vartheta_{p_s} = \sum_{p \in \overline{\mathcal{N}}} \sum_{\substack{\text{Ends}(\gamma_i) = (\tilde{\varphi}(p_i), Q_p), i=1, \dots, s \\ \pi_K(v_{\gamma_1} + \dots + v_{\gamma_s}) = p}}^{\gamma_1, \dots, \gamma_s} c_{\gamma_1} \cdots c_{\gamma_s} z^{v_{\gamma_1} + \dots + v_{\gamma_s} - \tilde{\varphi}(p)} \vartheta_p.$$

We denote the coefficient of ϑ_p by $\alpha_K(p_1, \dots, p_s; p) \in \mathbb{k}_q((K))_{\mathcal{I}_K}$, where $\mathcal{I}_K := \mathcal{I} \cap K$. Thus,

$$\alpha_K(p_1, \dots, p_s; p) = \sum_{k \in K} \alpha(\tilde{\varphi}(p_1), \dots, \tilde{\varphi}(p_s), \tilde{\varphi}(p) + k) z^k.$$

Similarly, if $\tilde{\varphi}(\overline{\mathcal{N}})$ and $P_K := P \cap K$ together generate the monoid P , we can express \widehat{B}_P as a $\mathbb{k}_q[P_K]$ -algebra generated by the same theta functions with the same multiplication rule (structure constants now contained in $\mathbb{k}_q[[P_K]]_{\mathcal{I}_K}$).

Remark 2.9. \widehat{B} does not depend on the choice of P : Products of infinite sums of theta functions in \widehat{B} can be handled equivalently using the filtration induced by the function d from Equation 6 (i.e., multiply ϑ_p 's with lower $d(p)$ first) instead of the one induced by \mathcal{I} .

Remark 2.10. Let $\overline{\mathcal{M}} := \pi_1(\mathcal{N}) \subset \mathcal{M}_{\mathbb{Q}}$. One can easily check that if $Q \in \overline{\mathcal{M}}_{\mathbb{R}}$, then for any $p \in \mathcal{N}$, any broken line with ends (p, Q) is entirely contained in $\overline{\mathcal{M}}_{\mathbb{R}}$. Thus, we can always restrict to $\overline{\mathcal{M}}_{\mathbb{R}}$, rather than dealing with all of $\mathcal{M}_{\mathbb{R}}$.

Furthermore, we can think of the scattering diagram as living in $\mathcal{N}_{\mathbb{R}}$ by, for each wall $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$, taking $(\pi_1^{-1}(\mathfrak{d} \cap \overline{\mathcal{M}}_{\mathbb{R}}), g_{\mathfrak{d}})$ as a wall with support in $\mathcal{N}_{\mathbb{R}}$. Broken lines and theta functions are constructed in essentially the same way, except that now for each broken line γ , we require that $-\gamma'$ is equal to the exponent of the attached monomial instead of π_1 of the exponent. This is the perspective primarily used in [GHKK14]. We will use this viewpoint in §5.4 to understand the middle cluster algebra.

2.5. Nondegeneracy of the trace pairings. Consider the maps

$$\text{Tr} : \mathbb{k}_q\langle\langle\mathcal{N}\rangle\rangle \rightarrow \mathbb{k}_q, \quad \sum_{p \in \mathcal{N}} a_p \vartheta_p \mapsto a_0$$

and

$$\text{Tr}_K : \mathbb{k}_q\langle\langle P \rangle\rangle \rightarrow \mathbb{k}_q[[P_K]]_{\mathcal{I}_K}, \quad \sum_{p \in \overline{\mathcal{N}}} a_p \vartheta_p \mapsto a_0.$$

These induce s -point functions

$$\text{Tr}^s : \widehat{B}^s \rightarrow \mathbb{k}_q, \quad (f_1, \dots, f_s) \mapsto \text{Tr}(f_1 \cdots f_s),$$

and similarly for $\text{Tr}_K^s : \widehat{B}_P^s \rightarrow \mathbb{k}_q[[P_K]]_{\mathcal{I}_K}$. The following theorem implies that these uniquely determine the multiplication structure of \widehat{B} and \widehat{B}_P . A more technical algebro-geometric proof of the claim for certain Tr_K in the two-dimensional classical situation has previously been found in [GHK].

Theorem 2.11. *Equip $\mathbb{k}_q\langle\langle\mathcal{N}\rangle\rangle$ and $\mathbb{k}_q\langle\langle\mathcal{P}\rangle\rangle$ with the algebra structures induced by \widehat{B} and \widehat{B}_P , respectively. The maps*

$$\text{Tr}^\vee : \mathbb{k}_q\langle\langle\mathcal{N}\rangle\rangle \rightarrow \text{Hom}_{\mathbb{k}_q}(\widehat{B}, \mathbb{k}_q), \quad a \mapsto [b \mapsto \text{Tr}(ab)]$$

and

$$\mathrm{Tr}_K^\vee : \mathbb{k}_q \langle\langle P \rangle\rangle \rightarrow \mathrm{Hom}_{\mathbb{k}_q \llbracket P_K \rrbracket}(\widehat{B}_P, \mathbb{k}_q \llbracket P_K \rrbracket), \quad a \mapsto [b \mapsto \mathrm{Tr}_K(ab)]$$

are injective. Thus, this associative \mathbb{k}_q - (resp. $\mathbb{k}_q \llbracket P_K \rrbracket$ -) algebra structure on $\mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$ (resp. $\mathbb{k}_q \langle\langle P \rangle\rangle$) is the unique such structure compatible with the topological \mathbb{k}_q -module structure (and the $\mathbb{k}_q \llbracket P_K \rrbracket$ -module structure)³ for which the corresponding 2- and 3-point functions on $\mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$ (resp. $\mathbb{k}_q \langle\langle P \rangle\rangle$) are those of \widehat{B} (resp. \widehat{B}_P).

In other words, specifying the structure constants of the form $\alpha(p_1, p_2; 0)$ and $\alpha(p_1, p_2, p_3; 0)$ determines all the other structure constants α , and similarly with α_K .

Proof. For the Tr case, note that we can express any $f \in \mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$ as $f_Q = \sum_{p \in S} a_p \vartheta_{p,Q} \in \mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$, where $S \subset \mathcal{N}$ is a subset such that $a_p \in \mathbb{k}_q \setminus \{0\}$ for each $p \in S$. Choose $p_0 \in S$ such that $d(p_0) \leq d(p)$ for all $p \in S$. When a broken line bends, d of the attached monomial increases. The z^{p_0} -coefficient of $f_Q \in \mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$ is therefore a_{p_0} , with this term corresponding to the straight broken line with ends (p_0, Q) . Similarly, $\vartheta_{-p_0, Q} = z^{-p_0} + \sum_{r \in S'} b_r z^r$, where the z^{-p_0} -term corresponds to the straight broken line with ends $(-p_0, Q)$, and $d(r) > d(p_0)$ for each $r \in S'$. We thus see that $\mathrm{Tr}(f \vartheta_{-p_0}) = a_{p_0} \neq 0$.

The argument for Tr_K is similar. In place of $-p_0$ (which is typically not in P), we use any $p_1 \in P$ such that $p_0 + p_1 \in P_K$. Then the coefficient of $\vartheta_{p_0+p_1} = z^{p_0+p_1}$ in $\mathrm{Tr}_K(f \vartheta_{p_1})$ is $a_{p_0} \neq 0$.

For the final claims, suppose we want to describe the product of two elements $a, b \in \mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$. The above injectivity implies that it is enough to specify $\mathrm{Tr}(abc) = \mathrm{Tr}^2(ab, c)$ for each $c \in \mathbb{k}_q \langle\langle \mathcal{N} \rangle\rangle$, and this is equal to $\mathrm{Tr}^3(a, b, c)$. Similarly for $\mathbb{k}_q \langle\langle P \rangle\rangle$ with Tr_K . \square

Remark 2.12. Recall that a Frobenius R -algebra is defined to be an R -algebra A , together with an R -algebra homomorphism $\mathrm{Tr} : A \rightarrow R$, such that the map $\mathrm{Tr}^\vee : A \rightarrow \mathrm{Hom}_R(A, R)$, $a \mapsto [b \mapsto \mathrm{Tr}(ab)]$, is an isomorphism. This forces A to be finite-dimensional. If we allow Tr^\vee to instead be just injective, rather than an isomorphism, we could define **infinite dimensional Frobenius algebras**. Then Theorem 2.11 says that Tr and Tr_K make \widehat{B} and \widehat{B}_P into infinite dimensional Frobenius \mathbb{k}_q - and $\mathbb{k}_q \llbracket P_K \rrbracket$ -algebras, respectively.

3. FROM BROKEN LINES TO TROPICAL CURVES

3.1. Standard initial scattering diagrams. We continue with the notation and setup of the previous sections.

For each $n \in \mathcal{N}^+$, we have an element $\mathrm{Li}_2(-z^n; q) \in \widehat{\mathfrak{g}}$, where

$$\mathrm{Li}_2(x; q) := \sum_{k=1}^{\infty} \frac{x^k}{k(q^k - q^{-k})}$$

is the quantum dilogarithm. Let

$$(13) \quad \Psi_q(z^{f_i}) := \exp(-\mathrm{Li}_2(-z^{f_i}; q)) = \prod_{a=1}^{\infty} \frac{1}{1 + q^{2a-1} z^{f_i}} \in G.$$

One checks that conjugation by $\Psi_{q^{1/d_i}}(z^{f_i})$ satisfies:

$$(14) \quad \Psi_{q^{1/d_i}}(z^{f_i})^{-1} z^n \Psi_{q^{1/d_i}}(z^{f_i}) = \left(\prod_{a=1}^{|d_i| \mathbf{W}(n, f_i)} \left(1 + q^{\mathrm{sgn}(\mathbf{W}(n, f_i))(2a-1)/d_i} z^{f_i} \right)^{\mathrm{sgn}(\mathbf{W}(n, f_i))} \right) z^n.$$

³Compatibility with the module structures just means that \mathbb{k}_q or $\mathbb{k}_q \llbracket P_K \rrbracket$ act on the algebras the way we would expect. The topological or filtered structure is just needed to determine how we deal with products of infinite sums.

In particular, the $q^{1/D} \mapsto 1$ limit is

$$(15) \quad z^n \mapsto z^n (1 + z^{f_i})^{d_i \mathbf{W}(n, f_i)}$$

For the rest of the paper, our initial scattering diagrams will all be of the form

$$(16) \quad \mathfrak{D}_{\text{in}} := \{(f_i^\perp, \Psi_{q^{1/d_i}}(z^{f_i})) | i \in I \setminus F\}.$$

3.2. Tropical curves and tropical disks.

Notation 3.1. For any weighted graph Γ , possibly with some 1-valent vertices removed, we let $\Gamma^{[0]}$, $\Gamma^{[1]}$, $\Gamma_\infty^{[0]}$, and $\Gamma_\infty^{[1]}$ denote the vertices, edges, 1-valent vertices, and non-compact edges, respectively. We denote the weight function of Γ by $w_\Gamma : \Gamma^{[1]} \rightarrow \mathbb{Z}_{\geq 1}$.

We now define marked tropical curves and disks. Let $\bar{\Gamma}$ be a weighted, connected, finite tree. We mark vertices of $\bar{\Gamma}$ with a map $\mu : \{1, \dots, m\} \rightarrow \Gamma^{[0]}$, $\mu(i) \in \Gamma^{[0]} \setminus \Gamma_\infty^{[0]}$ for $i \neq 1$ (i.e., only $i = 1$ can map to a univalent vertex). We require all divalent vertices to be **marked** (i.e., in the image of μ). We will in fact almost always have $m = 1$, and we assume this from now on unless otherwise specified (there is an $m = 0$ case in §3.6). Denote $\mu(1)$ by Q_{out} . Now let

$$\Gamma := (\bar{\Gamma} \setminus \Gamma_\infty^{[0]}) \cup Q_{\text{out}}$$

That is, we remove all the 1-valent vertices, unless Q_{out} is 1-valent, in which case we remove all the 1-valent vertices except for Q_{out} . We mark the unbounded edges with a bijection $\epsilon : \{1, \dots, e_\infty\} \xrightarrow{\sim} \Gamma_\infty^{[1]}$.

Definition 3.2. A **parameterized marked tropical curve** in $\overline{\mathcal{M}}_{\mathbb{R}} := \pi_1(\mathcal{N}_{\mathbb{R}})$ (cf. Remark 2.10) is the data Γ, μ, ϵ as above, along with a proper map $h : \Gamma \rightarrow \overline{\mathcal{M}}_{\mathbb{R}}$ such that:

- For each $E \in \Gamma^{[1]}$, $h|_E$ is an embedding with image contained in an affine line of rational slope.
- The following “balancing condition” holds for every $V \in \Gamma^{[0]}$: Let $E_1, \dots, E_n \in \Gamma^{[1]}$ denote the edges containing V , and let $v'_i \in \overline{\mathcal{M}}$, $i = 1, \dots, n$, denote the primitive integral vector emanating from V in the direction $h(E_i)$. Let $v_i := w_\Gamma(E_i)v'_i$. Then

$$(17) \quad \sum_{i=1}^n v_i = 0.$$

Two parameterized marked tropical curves $h_i : \Gamma_i \rightarrow \overline{\mathcal{M}}_{\mathbb{R}}$, $i = 1, 2$, are isomorphic if there is a homeomorphism $\Phi : \Gamma_1 \rightarrow \Gamma_2$ respecting the weights and markings. A **marked tropical curve** is an isomorphism class of parameterized marked tropical curves. A **marked tropical disk** is defined in the same way, except that the balancing condition at Q_{out} is not required to hold. We will often drop the adjective “marked.”

We will abuse notation and let \mathbf{W} denote the nondegenerate skew-form on $\overline{\mathcal{M}}$ induced by \mathbf{W} on \mathcal{N} —that is, $\mathbf{W}(\pi_1(n), \pi_1(n')) := \mathbf{W}(n, n')$.

Definition 3.3. Let $h : \Gamma \rightarrow \overline{\mathcal{M}}_{\mathbb{R}}$ be a tropical curve or disk. Assume all vertices in $\Gamma^{[0]} \setminus Q_{\text{out}}$ are trivalent. For $V \in \Gamma^{[0]} \setminus Q_{\text{out}}$, let v_1, v_2, v_3 be the weighted tangent vectors of $h(\Gamma)$ emanating from $h(V)$, as in Equation 17. The Block-Göttsche multiplicity of V is

$$\text{Mult}_{V,q}(h) := [\mathbf{W}(v_1, v_2)]_q = [\mathbf{W}(v_2, v_3)]_q = [\mathbf{W}(v_3, v_1)]_q.$$

Here we use the notation $[w]_q$ as in (5). The equalities follow from the balancing condition. For Q_{out} , we again let v_1, \dots, v_s denote the weighted tangent vectors of $h(\Gamma)$ emanating from $h(Q_{\text{out}})$, but now

the **order** matters! (Our choice of order will depend on the order in which we multiply our quantum theta functions). We then define the quantum multiplicity of Q_{out} to be

$$\text{Mult}_{Q_{\text{out}},q}(h) := q^{\sum_{i<j} \mathbf{W}(v_i, v_j)}.$$

Here, if Q_{out} is 1-valent, then $\text{Mult}_{Q_{\text{out}},q}(h) := 1$. The **quantum multiplicity** of h is now defined as:

$$\text{Mult}_q(h) := \prod_{V \in \Gamma^{[0]}} \text{Mult}_{V,q}(h).$$

Taking the limits as $q^{1/D} \rightarrow 1$, we obtain the **classical multiplicity**: $\text{Mult}_{Q_{\text{out}}}(h) := 1$, for $V \neq Q_{\text{out}}$, $\text{Mult}_V(h) := |\mathbf{W}(v_1, v_2)|$, and $\text{Mult}(h) := \prod_{V \in \Gamma^{[0]}} \text{Mult}_V(h)$.

Remark 3.4. We note the following equivalent definitions of the multiplicities of vertices. For each $v \in \overline{\mathcal{M}}$ we let \tilde{v} denote any vector in \mathcal{N} such that $\pi_1(\tilde{v}) = v$. Assuming $\mathbf{W}(v_2, v_1) \geq 0$ (and reordering otherwise), we could equivalently define $\text{Mult}_{V,q}(h)$ by

$$[\hat{z}^{\tilde{v}_1}, \hat{z}^{\tilde{v}_2}] = \text{Mult}_{V,q}(h) \hat{z}^{\tilde{v}_1 + \tilde{v}_2}.$$

Similarly, we could define the multiplicity of Q_{out} by

$$(18) \quad z^{\tilde{v}_1} \dots z^{\tilde{v}_s} = \text{Mult}_{Q_{\text{out}},q}(h) z^{\tilde{v}_1 + \dots + \tilde{v}_s}.$$

By defining multiplicities in this way, we expect that the arguments of this section could be modified for more general \mathcal{N}^+ -graded Lie algebras \mathfrak{g} (see also Remark 2.6).

3.3. The main theorem. Let $\mathbf{p} := (\overline{p}_1, \dots, \overline{p}_s)$, be an s -tuple of non-zero vectors in $\overline{\mathcal{M}}$, $s \geq 0$. Let $\mathbf{m} := (m_i)_{i \in I \setminus F}$ be another tuple of vectors $m_i \in \overline{\mathcal{M}}$, this time indexed over $I \setminus F$. Let $\mathbf{w} := (\mathbf{w}_i)_{i \in I \setminus F}$ be a tuple of weight vectors $\mathbf{w}_i := (w_{i1}, \dots, w_{il_i})$ with $0 < w_{i1} \leq \dots \leq w_{il_i}$, $w_{ij} \in \mathbb{Z}$. Let

$$\text{Aut}(\mathbf{w}) \subset \prod_{i \in I \setminus F} S_{l_i}$$

be the group of automorphisms of the second indices of the \mathbf{w}_i 's which act trivially on \mathbf{w} . Denote $|\mathbf{w}_i| := \sum_j w_{ij}$. For $i \in I \setminus F$, $1 \leq j \leq l_i$, choose generic $m_{ij} \in \overline{\mathcal{M}}_{\mathbb{R}}$, and define codimension 1 subsets

$$\mathfrak{d}_{ij} := m_{ij} + (f_i^\perp \cap \overline{\mathcal{M}}_{\mathbb{R}}) \subset \overline{\mathcal{M}}_{\mathbb{R}}.$$

We let \mathbf{m}_{ij} denote the collection of all these choices of m_{ij} 's. Let Q be a point in $\overline{\mathcal{M}}_{\mathbb{R}}$, or if $s = 0$, a line.

Definition 3.5. Let $\mathfrak{T}_{\mathbf{m}, \mathbf{p}, \mathbf{w}}(\mathbf{m}_{ij}, Q)$ denote the set of marked tropical disks such that:

- The number of unbounded edges e_∞ equals $s + \sum_i l_i$. We re-index our edge-marking by replacing the domain of ϵ with $\{1, \dots, s\} \cup \{(i, j) | i \in I \setminus F, j = 1, \dots, l_i\}$, and we denote $F_k := \epsilon(k)$, $E_{ij} := \epsilon((i, j))$.
- For each (i, j) , $h(E_{ij}) \subset \mathfrak{d}_{ij}$ with m_i pointing in the unbounded direction.
- For each $1 \leq k \leq s$, \overline{p}_k points in the unbounded direction of $h(F_k)$.
- $w_\Gamma(E_{ij}) = w_{ij}|m_i|$, and $w_\Gamma(F_i) = |\overline{p}_k|$.
- Q is $\max(s, 1)$ -valent, and $h(Q_{\text{out}}) = Q$.

For generic choices of m_{ij} 's and Q , all vertices other than Q_{out} are trivalent, so we can compute their Block-Göttsche multiplicities. Recall that defining $\text{Mult}_q(Q_{\text{out}})$ requires **fixing an ordering** of the weighted tangent vectors v_1, \dots, v_s emanating from Q_{out} . Again for generic m_{ij} 's and Q , each component of $\Gamma \setminus Q_{\text{out}}$ will contain exactly one F_i (assuming $s \geq 1$). We then let v_i be the weighted

tangent vector corresponding to the component containing F_i (so the ordering of \mathbf{p} determines the order of the v_i 's). We can now define $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ to be the Block-Göttsche weighted count of marked tropical curves in $\mathfrak{T}_{\mathbf{m}, \mathbf{p}, \mathbf{w}}(\mathbf{m}_{ij}, Q)$. That is,

$$N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) := \sum_{h \in \mathfrak{T}_{\mathbf{m}, \mathbf{p}, \mathbf{w}}(\mathbf{m}_{ij}, Q)} \text{Mult}_q(h).$$

Remark 3.6. We have dropped the m_{ij} 's from the notation for the counts $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$. We will find in Theorem 3.12 that for tropical curves, these counts are indeed independent of the generic choices of m_{ij} 's. However, for tropical disks we must assume that the \mathfrak{d}_{ij} 's are close to the origin relative to Q .

For $n \in \mathcal{N}$, let $\mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)$ be the set of weight vectors \mathbf{w} as above such that

$$\sum_{i \in I \setminus F} \sum_{j=1}^{l_i} w_{ij} f_i + \sum_{k=1}^s p_k = n.$$

Note that $\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(0)$ exactly means that objects in $\mathfrak{T}_{\mathbf{m}, \mathbf{p}, \mathbf{w}}(\mathbf{m}_{ij}, Q)$ are tropical curves rather than tropical disks. Define

$$R_{w, d; q} := \frac{(-1)^{w-1}}{w[w/d]_q},$$

and

$$R_{\mathbf{w}} := \prod_{i, j} R_{w_{ij}, d_i; q}.$$

The goal of this section is to prove the following theorem. Here, we assume \mathfrak{D}_{in} is as in Equation 16. \widehat{B} and the theta functions are constructed with respect to $\mathfrak{D} := \text{Scat}(\mathfrak{D}_{\text{in}})$.

Theorem 3.7. *For $i \in I \setminus F$, take $m_i := \pi_1(f_i)$ to define \mathbf{m} . For $p_1, \dots, p_s \in \mathcal{N} \setminus \ker(\pi_1)$, $s \geq 1$, take $\overline{p_i} := \pi_1(p_i)$ to define \mathbf{p} . Let $Q \in \overline{\mathcal{M}}_{\mathbb{R}}$ be generic and sufficiently far from the origin (relative to the \mathfrak{d}_{ij} 's). For any $n \in \mathcal{N}$, the coefficient of z^n in the product $\vartheta_{p_1, Q} \cdots \vartheta_{p_s, Q}$ is*

$$(19) \quad \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)} N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|}$$

If Q is sufficiently far from the origin and close to the ray ρ_n generated by $\pi_1(n)$, or if $\mathbf{W}(n, \mathcal{N}^+) = 0$ and Q is any generic point (not necessarily far from the origin), then (19) gives the structure constant $\alpha(p_1, \dots, p_s; n)$.

3.4. Factored, perturbed, and asymptotic scattering diagrams. We closely follow some constructions from §1.4 of [GPS10], modified for our situation.

Definition 3.8. The **asymptotic scattering diagram** \mathfrak{D}_{as} of \mathfrak{D} is defined as follows: Every wall $(m_0 + \mathfrak{d}_0, g) \in \mathfrak{D}$, with \mathfrak{d}_0 an $(r-1)$ -dimensional convex (but not necessarily strictly convex) rational polyhedral cone and $m_0 \in \mathcal{M}_{\mathbb{R}}$, is replaced by the wall (\mathfrak{d}_0, g) .

Intuitively, \mathfrak{D}_{as} is obtained from \mathfrak{D} by zooming very far away from $\mathcal{M}_{\mathbb{R}}$. Note that

$$(20) \quad \text{Scat}(\mathfrak{D}_{\text{as}}) = (\text{Scat}(\mathfrak{D}))_{\text{as}}.$$

We will use the technique from [GPS10] in which one factors an initial scattering diagram \mathfrak{D}_{in} , deforms the factored scattering diagram by moving the supports of the initial walls, constructs Scat of the deformed scattering diagram, and then takes the asymptotic scattering diagram to obtain $\text{Scat}(\mathfrak{D}_{\text{in}})$.

Let $T := \mathbb{k}_q[t_i | i \in I]$ (commutative), $T_k := T / \langle t_i^{k+1} | i \in I \setminus F \rangle$. For $p := \sum a_i f_i \in \mathcal{N}$, we denote $t^p := \prod_{i \in I} t_i^{a_i}$. Let \mathfrak{D}_{in} be the initial scattering diagram over $G \widehat{\otimes} T$ in $\mathcal{M}_{\mathbb{R}}$ given by

$$\mathfrak{D}_{\text{in}} := \{ (f_i^\perp, \Psi_{q^{1/d_i}}(t_i z^{f_i})) | i \in I \setminus F \}.$$

Let $\mathfrak{D} := \text{Scat}(\mathfrak{D}_{\text{in}})$. We can use this \mathfrak{D} to obtain theta functions in $T((\mathcal{N}))$ —the only non-obvious modification to Definition 2.7 is that the initial monomials for broken lines contributing to $\vartheta_{p,Q} \in T((\mathcal{N}))$ are taken to be $t^p z^p$.

Remark 3.9. One notes that every instance of t^p in this section is multiplied by the corresponding z^p . Thus, the ring generated by $\{\vartheta_{p,Q}\}_{p \in \mathcal{N}} \subset T((\mathcal{N}))$ is isomorphic to the corresponding subring of $\mathbb{k}_q((\mathcal{N}))$ by mapping $t^p z^p \mapsto z^p$. As in Remark 2.9, our precise choice of P is unimportant for understanding \widehat{B} , so we might as well take P to be the positive span of the f_i 's. Then t^p vanishing in T_k is equivalent to z^p vanishing modulo $k\mathcal{I}$, so we similarly have an identification of the ring generated by $\{\vartheta_{p,Q}\}_{p \in P \setminus k\mathcal{I}} \subset T_k((\mathcal{N}))$ with B_k . That is, $\vartheta_{p,Q} \in B_k$ can be identified with $\vartheta_{p,Q}^{\text{Scat}_k(\mathfrak{D})} \in T_k((\mathcal{N}))$ for $\text{Scat}_k(\mathfrak{D})$ as defined below. Note that, for fixed \mathbf{p} and n , it suffices to check (19) in B_k for k sufficiently large. This is what we will do.

Let $\beta_k : T \rightarrow T_k$ denote the projection. Define

$$\text{Scat}_k(\mathfrak{D}) := \beta_k(\text{Scat}(\mathfrak{D})) = \text{Scat}(\beta_k(\mathfrak{D})).$$

By β_k of a scattering diagram, we mean that β_k is applied to $g_{\mathfrak{d}}$ for every wall $(\mathfrak{d}, g_{\mathfrak{d}})$ in the scattering diagram. Define the inclusion

$$\begin{aligned} \iota : T_k &\hookrightarrow \widetilde{T}_k := \mathbb{k}_q[u_{ij} | i \in I \setminus F, 1 \leq j \leq k] / \langle u_{ij}^2 | i \in I \setminus F, 1 \leq j \leq k \rangle \\ t_i &\mapsto \sum_{j=1}^k u_{ij}. \end{aligned}$$

Recall the notation $\hat{z}^n := \frac{z^n}{q^n - 1}$ from §2.2. We have

$$(21) \quad \log \Psi_{q^{1/d_i}}(t_i z^{f_i}) = -\text{Li}_2(-t_i z^{f_i}; q^{1/d_i}) = \sum_{w \geq 1} \frac{(-1)^{w-1} z^{wf_i} t_i^w}{w(q^{w/d_i} - q^{-w/d_i})} = \sum_{w \geq 1} R_{w,d_i;q} \hat{z}^{wf_i} t_i^w.$$

Applying $\iota \circ \beta_k$, we get

$$\log \Psi_{q^{1/d_i}}(t_i z^{f_i}) = \sum_{w=1}^k \sum_{\#J=w} w! R_{w,d_i;q} \hat{z}^{wf_i} u_{iJ},$$

where the second sum is over all subsets $J \subset \{1, \dots, k\}$ of size w , and

$$u_{iJ} := \prod_{j \in J} u_{ij}.$$

Exponentiating gives the factorization

$$\Psi_{q^{1/d_i}}(t_i z^{f_i}) = \prod_{w=1}^k \prod_{\#J=w} (1 + w! R_{w,d_i;q} \hat{z}^{wf_i} u_{iJ}).$$

We can thus factor and perturb the scattering diagram $\beta_k(\mathfrak{D}_{\text{in}})$ to get

$$(22) \quad \widetilde{\mathfrak{D}}_k^0 := \{(\mathfrak{d}_{iwJ}, 1 + w! R_{w,d_i;q} \hat{z}^{wf_i} u_{iJ}^J) | 1 \leq w \leq k, J \subset \{1, \dots, k\}, \#J = w\},$$

where $\mathfrak{d}_{iwJ} := f_i^\perp + m_{iwJ}$ for some generic collection of m_{iwJ} 's in $\overline{\mathcal{M}}_{\mathbb{R}}$.

As in [GPS10, §1.4], we produce a sequence of scattering diagrams $\tilde{\mathfrak{D}}_k^0, \tilde{\mathfrak{D}}_k^1, \tilde{\mathfrak{D}}_k^2, \dots, \tilde{\mathfrak{D}}_k^i, \dots$ in which $\tilde{\mathfrak{D}}_k^i = \text{Scat}(\tilde{\mathfrak{D}}_k^0)$ for $i > k\#(I \setminus F)$. Assume inductively that:

- (a) Each wall in $\tilde{\mathfrak{D}}_k^i$ is of the form $(\mathfrak{d}, 1 + a_{\mathfrak{d}} \hat{z}^{n_{\mathfrak{d}}} u_{I_{\mathfrak{d}}})$, where $a_{\mathfrak{d}} \in \mathbb{k}_q$, $n_{\mathfrak{d}} \in \mathcal{N}$, $I_{\mathfrak{d}}$ is a subset of $(I \setminus F) \times \{1, \dots, k\}$, and

$$u_{I_{\mathfrak{d}}} := \prod_{(i,j) \in I_{\mathfrak{d}}} u_{ij}.$$

- (b) There is no set W of walls in $\tilde{\mathfrak{D}}_k^i$ of cardinality ≥ 3 such that $\bigcap_{\mathfrak{d} \in W} \mathfrak{d}$ has codimension ≥ 2 and $I_{\mathfrak{d}_1} \cap I_{\mathfrak{d}_2} = \emptyset$ for each pair of distinct $\mathfrak{d}_1, \mathfrak{d}_2 \in W$. Note that two walls $\mathfrak{d}_1, \mathfrak{d}_2$ only produce a new wall as in Lemma 2.5 if $I_{\mathfrak{d}_1} \cap I_{\mathfrak{d}_2} = \emptyset$.

These conditions clearly hold for $\tilde{\mathfrak{D}}_k^0$. To get $\tilde{\mathfrak{D}}_k^i$ from $\tilde{\mathfrak{D}}_k^{i-1}$, consider each pair $\mathfrak{d}_1, \mathfrak{d}_2 \in \tilde{\mathfrak{D}}_k^{i-1}$ which satisfies:

- (i) $\{\mathfrak{d}_1, \mathfrak{d}_2\} \not\subseteq \tilde{\mathfrak{D}}_k^{i-2}$,
- (ii) $\mathfrak{d}_1 \cap \mathfrak{d}_2 \neq \emptyset$ has codimension 2 and is not contained in the boundary of either \mathfrak{d}_1 or \mathfrak{d}_2 ,
- (iii) $I_{\mathfrak{d}_1} \cap I_{\mathfrak{d}_2} = \emptyset$.

Writing $f_{\mathfrak{d}_i} = 1 + a_{\mathfrak{d}_i} \hat{z}^{n_{\mathfrak{d}_i}} u_{I_{\mathfrak{d}_i}}$, Lemma 2.5 suggests a new wall:

$$(23) \quad \mathfrak{d}(\mathfrak{d}_1, \mathfrak{d}_2) := ((\mathfrak{d}_1 \cap \mathfrak{d}_2) + \mathbb{R}_{\leq 0}[\pi_1(n_{\mathfrak{d}_1} + n_{\mathfrak{d}_2})], 1 + a_{\mathfrak{d}_1} a_{\mathfrak{d}_2} [\|\mathbf{W}(n_{\mathfrak{d}_1}, n_{\mathfrak{d}_2})\|]_q \hat{z}^{n_1 + n_2} u_{I_{\mathfrak{d}_1} \cup I_{\mathfrak{d}_2}}).$$

We now define

$$\tilde{\mathfrak{D}}_k^i := \tilde{\mathfrak{D}}_k^{i-1} \cup \{\mathfrak{d}(\mathfrak{d}_1, \mathfrak{d}_2) \mid \mathfrak{d}_1, \mathfrak{d}_2 \text{ satisfying (i)-(iii) above}\}.$$

Definition 3.10. If $\mathfrak{d} = \mathfrak{d}(\mathfrak{d}_1, \mathfrak{d}_2)$, define $\text{Parents}(\mathfrak{d}) := \{\mathfrak{d}_1, \mathfrak{d}_2\}$. Recursively define $\text{Ancestors}(\mathfrak{d})$ by $\text{Ancestors}(\mathfrak{d}) := \{\mathfrak{d}\} \cup \bigcup_{\mathfrak{d}' \in \text{Parents}(\mathfrak{d})} \text{Ancestors}(\mathfrak{d}')$. Define

$$\text{Leaves}(\mathfrak{d}) := \{\mathfrak{d}' \in \text{Ancestors}(\mathfrak{d}) \mid \mathfrak{d}' \text{ is the support of a wall in } \tilde{\mathfrak{D}}_k^0\}.$$

It is clear that $\tilde{\mathfrak{D}}_k^i$ satisfies inductive hypothesis (a). For hypothesis (b), suppose we do have such a bad set of walls W . Since the $I_{\mathfrak{d}_i}$'s are pairwise disjoint, the sets $\text{Leaves}(\mathfrak{d}_i)$ are also disjoint. Thus, slightly shifting the m_{iwJ} 's will shift the walls in W independently, and so we can avoid having this bad set W by choosing the m_{iwJ} 's more generically.

Since the cardinality of $I_{\mathfrak{d}}$ for the new walls increases with each step and is bounded by $k\#(I \setminus F)$, we see that the process stabilizes to a scattering diagram $\tilde{\mathfrak{D}}_k^\infty$ once i reaches $k\#(I \setminus F)$. Furthermore, since the wall-crossing automorphisms $\theta_{\mathfrak{d}_1}$ and $\theta_{\mathfrak{d}_2}$ commute for $I_{\mathfrak{d}_1} \cap I_{\mathfrak{d}_2} \neq \emptyset$, Lemma 2.5 implies that $\tilde{\mathfrak{D}}_k^\infty$ is consistent. Thus, we have

$$\text{Scat}_k(\mathfrak{D}_{\text{in}}) = (\tilde{\mathfrak{D}}_k^\infty)_{\text{as}}.$$

Now, as in Definition 3.5, fix a weight vector $\mathbf{w} := (\mathbf{w}_i)_{i \in I \setminus F}$, $\mathbf{w}_i := (w_{i1}, \dots, w_{il_i})$, generic vectors $m_{ij} \in \overline{\mathcal{M}}_{\mathbb{R}}$ for $i \in I \setminus F$, $1 \leq j \leq l_i$, and define $\mathfrak{d}_{ij} := f_i^\perp + m_{ij}$. For k sufficiently large, we can associate each m_{ij} with a subset $J_{ij} \subset \{1, \dots, k\}$, $\#J_{ij} = w_{ij}$, such that $J_{ij} \cap J_{i'j'} = \emptyset$ for $j \neq j'$. Take $\tilde{\mathfrak{D}}_k^0$ as in (22) with $\mathfrak{d}_{iw_{ij}J_{ij}}$ taken to be \mathfrak{d}_{ij} . Let $Q \in \overline{\mathcal{M}}_{\mathbb{R}}$ be generic (i.e., not contained in any joint of $\tilde{\mathfrak{D}}_k^\infty$). The following is a higher-dimensional quantum version of Theorem 2.4 from [GPS10] (i.e., a higher dimensional version of Lemmas 4.5 and 4.6 in [FS15]).

Lemma 3.11. *There exists a tropical disk $h : \Gamma \rightarrow \overline{\mathcal{M}}_{\mathbb{R}}$ in $\mathfrak{T}_{\mathbf{m}, \emptyset, \mathbf{w}}(\mathbf{m}_{ij}, Q)$ if and only if Q is in the support of some $(\mathfrak{d}, 1 + a_{\mathfrak{d}} \hat{z}^{n_{\mathfrak{d}}} u_{I_{\mathfrak{d}}}) \in \tilde{\mathfrak{D}}_k^{\infty}$ with $\text{Leaves}(\mathfrak{d}) = \bigcup_{i,j} \{\mathfrak{d}_{iw_{ij} J_{ij}}\}$. This h is unique if $Q \notin \text{Joints}(\tilde{\mathfrak{D}}_k^{\infty})$. Furthermore, $I_{\mathfrak{d}} = \bigcup_{\mathfrak{d}_{iw_{ij} J_{ij}} \in \text{Leaves}(\mathfrak{d}), l \in J_{ij}} (i, l)$,*

$$n_{\mathfrak{d}} = \sum_{i \in I \setminus F} \sum_{j=1}^{l_i} w_{ij} f_i,$$

and

$$(24) \quad a_{\mathfrak{d}} = \text{Mult}_q(h) R_{\mathbf{w}} \prod_{i,j} (w_{ij}!).$$

Proof. The proof of [GPS10, Theorem 2.4] is easily modified to prove this Proposition. The idea is to construct the tropical disk by tracing away from Q in the direction $\pi_1(n_{\mathfrak{d}})$ until hitting a point $p \in \mathfrak{d}_1 \cap \mathfrak{d}_2$, where $\{\mathfrak{d}_1, \mathfrak{d}_2\} = \text{Parents}(\mathfrak{d})$. The resulting segment is given weight $|n_{\mathfrak{d}}|$. From p , extend the tropical disk in the directions $\pi_1(n_{\mathfrak{d}_1})$ and $\pi_1(n_{\mathfrak{d}_2})$ with weights $|n_{\mathfrak{d}_1}|$ and $|n_{\mathfrak{d}_2}|$, respectively. The balancing condition at p follows easily from (23). The process is repeated for each of these branches, and continues until every branch extends to infinity in some leaf. This gives the desired tropical disk. (24) is easily checked using (22) and (23). \square

3.5. Proof of Theorem 3.7 and the invariance of $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$. We give the details for the $s = 1$ case. The general s situation is similar. We wish to describe the z^n coefficient of $\vartheta_{p,Q} := \vartheta_{p,Q}^{\tilde{\mathfrak{D}}_k^{\infty}}$, for fixed $k > 0$, in terms of tropical curve counts. We can assume that Q is far enough from the origin for the z^n coefficient of $\vartheta_{p,Q}^{\tilde{\mathfrak{D}}_k^{\infty}}$ to agree with that of $\vartheta_{p,Q}^{(\tilde{\mathfrak{D}}_k^{\infty})_{\text{as}}} = \vartheta_{p,Q}^{\text{Scat}_k(\mathfrak{D}_{\text{in}})}$.

Let $z^{\mathbf{w}} := z^{\sum_i |\mathbf{w}_i| f_i}$, $z^{p+\mathbf{w}} := z^{p+\sum_i |\mathbf{w}_i| f_i}$, and similarly for $t^{\mathbf{w}}$ and $t^{p+\mathbf{w}}$. Take $\mathbf{p} := (\pi_1(p))$. We want to show that the coefficient of z^n in $\vartheta_{p,Q} \in \tilde{T}_k$ is

$$(25) \quad \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)} \left[N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) R_{\mathbf{w}} t^p \frac{1}{|\text{Aut}(\mathbf{w})|} \prod_{i \in I} \left(|\mathbf{w}_i|! \sum_{\substack{J_i \subset \{1, \dots, k\} \\ \#J_i = |\mathbf{w}_i|}} \prod_{j \in J_i} u_{ij} \right) \right].$$

Then since

$$t_i^{|\mathbf{w}_i|} = |\mathbf{w}_i|! \sum_{\substack{J_i \subset \{1, \dots, k\} \\ \#J_i = |\mathbf{w}_i|}} \prod_{j \in J_i} u_{ij},$$

this becomes

$$\sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)} N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|} t^n,$$

as desired (this is related to (19) using Remark 3.9).

Consider a broken line γ contributing to $\vartheta_{p,Q}$. For any wall $\mathfrak{d} \in \tilde{\mathfrak{D}}_k^{\infty}$ along which γ breaks at a point $Q_{\mathfrak{d}}$, we glue to γ the corresponding tropical disk $h_{Q_{\mathfrak{d}}}$ with endpoint $Q_{\mathfrak{d}}$ described in Lemma 3.11. Note that $h_{Q_{\mathfrak{d}}}$ together with γ (weighted by the indices of the exponents of the attached monomials) satisfies the balancing condition at $Q_{\mathfrak{d}}$, so repeating this for every break of γ results in a tropical disk h_{γ} . Define

$$\text{Leaves}(\gamma) = \bigcup_{Q_{\mathfrak{d}}} \text{Leaves}(\mathfrak{d}),$$

where the union is over all points $Q_{\mathfrak{d}}$ where γ bends. Let $\mathbf{w}^\gamma = (\mathbf{w}_i^\gamma | i \in I \setminus F)$, $\mathbf{w}_i^\gamma := (w_{i1}^\gamma, \dots, w_{il_i}^\gamma)$, be the weight-vector such that

$$\text{Leaves}(\gamma) = \{\mathfrak{d}_{iw_{ij}^\gamma J_{ij}} | i \in I \setminus F, 1 \leq j \leq l_i^\gamma\}$$

for some collection of subsets $J_{ij} \subset \{1, \dots, k\}$ with $\#J_{ij} = w_{ij}^\gamma$. Here, if $w_{ij}^\gamma < w_{ij'}^\gamma$, then $j < j'$, and if $w_{ij}^\gamma = w_{ij'}^\gamma$, then we use some fixed ordering on subsets of $\{1, \dots, k\}$ to determine the ordering for the weights. Let $J(\gamma) := \bigcup_{i,j} (i, J_{ij})$. Define \mathbf{m}_{ij}^γ by $m_{ij}^\gamma := m_{iw_{ij}^\gamma J_{ij}}$. Then one easily sees by unraveling the definitions that

$$h_\gamma \in \mathfrak{T}_{\mathbf{m}, \mathbf{p}}^{\text{trop}}(\mathbf{w}^\gamma, \mathbf{m}_{ij}^\gamma, Q).$$

Conversely, one sees that every tropical disk in $\mathfrak{T}_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(\mathbf{m}_{ij}, Q)$, for any fixed \mathbf{m}_{ij} corresponding to walls of $\tilde{\mathfrak{D}}_k^0$ and $\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)$, can be constructed in this way from some unique broken line contributing to the z^n coefficient of $\vartheta_{p, Q}$.

We now examine the monomial attached to γ . Recall that the initial monomial is $t^p z^p$. When γ breaks at a point $Q_{\mathfrak{d}}$ in a wall $(\mathfrak{d}, g_{\mathfrak{d}} = 1 + a_{\mathfrak{d}} \hat{z}^{n_{\mathfrak{d}}} u_{I_{\mathfrak{d}}})$, Lemma 2.8 tells us that the attached monomial changes from some $ct^p z^v$ to

$$a_{\mathfrak{d}} c \text{Mult}_{Q_{\mathfrak{d}}, q}(h_\gamma) u_{I_{\mathfrak{d}}} t^p z^{v+n_{\mathfrak{d}}}.$$

Combining this with the description of $a_{\mathfrak{d}} \hat{z}^{n_{\mathfrak{d}}} u_{I_{\mathfrak{d}}}$ in Lemma 3.11 and working inductively, one easily sees that the final monomial attached to γ is

$$(26) \quad c_\gamma z^{n_\gamma} = \text{Mult}_q(h_\gamma) R_{\mathbf{w}^\gamma} \prod_{i,j} (w_{ij}^\gamma!) t^p u_{J(\gamma)} z^{p+\mathbf{w}^\gamma}.$$

Adding up the final monomials for all γ with ends (p, Q) , final exponent n , and fixed $\text{Leaves}(\gamma)$ (hence fixed \mathbf{w}^γ , \mathbf{m}_{ij}^γ , and $J(\gamma)$), we get

$$(27) \quad N_{\mathbf{m}, \mathbf{p}}^{\text{trop}}(\mathbf{w}^\gamma, Q) R_{\mathbf{w}^\gamma} \prod_{i,j} (w_{ij}^\gamma!) t^p u_{J(\gamma)} z^n.$$

A priori, the $N_{\mathbf{m}, \mathbf{p}}^{\text{trop}}(\mathbf{w}^\gamma, Q)$ term here could depend on \mathbf{m}_{ij}^γ , which would prevent us from being able to combine this in a nice way with the other possible choices for $\text{Leaves}(\gamma)$. We therefore need the following theorem:

Theorem 3.12. *Let $\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(n)$. For $\mathbf{p} \neq \emptyset$, the Block-Göttsche counts $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ do not depend on the generic choice of \mathbf{m}_{ij} as long as Q is sufficiently far from the origin relative to the \mathfrak{d}_{ij} 's. If $\mathbf{W}(n, f_i) = 0$ for each $i \in I \setminus F$ (which in particular holds for tropical curves), then $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ is also independent of the generic choice of Q .*

Proof. For the $s = 1$ case, consider the situation above, where we saw that (27) gives the contribution to the z^n term of $\vartheta_{p, Q}$ coming from broken lines γ with fixed $\text{Leaves}(\gamma)$. We define a new initial scattering diagram $\tilde{\mathfrak{D}}_k^\gamma \subset \tilde{\mathfrak{D}}_k^0$ whose walls are the leaves of γ . Now observe that the broken lines determining the z^n term in the theta function $\vartheta_{p, Q}^{\text{Scat}(\tilde{\mathfrak{D}}_k^\gamma)}$ are exactly those whose final monomials we said sum to give (27). Thus, $N_{\mathbf{m}, \mathbf{p}}^{\text{trop}}(\mathbf{w}^\gamma, Q)$ is determined by $\vartheta_{p, Q}^{\text{Scat}(\tilde{\mathfrak{D}}_k^\gamma)}$. The choice of \mathbf{m}_{ij} does not affect the z^n term of $\vartheta_{p, Q}^{\text{Scat}(\tilde{\mathfrak{D}}_k^\gamma)}$ as long as Q is far enough from the origin for the z^n term of $\vartheta_{p, Q}^{\text{Scat}(\tilde{\mathfrak{D}}_k^\gamma)}$ to equal that of $\vartheta_{p, Q}^{\text{Scat}(\tilde{\mathfrak{D}}_k^\gamma)_{\text{as}}}$. This gives the result for $s = 1$. The higher s case is similar.

If $\mathbf{W}(n, f_i) = 0$, then path-ordered products act trivially on z^n , and so the independence from Q follows from [CPS]'s result mentioned in §2.4 about path-ordered products respecting the theta functions. \square

We can now sum over all possible choices for $\text{Leaves}(\gamma)$. Note that for each $J(\gamma)$, there are

$$(28) \quad \left(\prod_i \frac{|\mathbf{w}_i|!}{\prod_j w_{ij}!} \right) \left(\frac{1}{|\text{Aut } \mathbf{w}_\gamma|} \right)$$

ways to partition $J(\gamma)$ into subsets of the form (i, J_{ij}) , $\#J_{ij} = w_{ij}$ —these correspond to the different choices of $\text{Leaves}(\gamma)$ giving the same $J(\gamma)$. We multiply (27) by (28) to get the contribution from broken lines with fixed $J(\gamma)$ and \mathbf{w}^γ :

$$N_{\mathbf{m}, \mathbf{p}}^{\text{trop}}(\mathbf{w}^\gamma, Q) \frac{R_{\mathbf{w}^\gamma}}{|\text{Aut } \mathbf{w}^\gamma|} \prod_i (|\mathbf{w}_i^\gamma|!) t^p u_{J(\gamma)} z^n.$$

Then we sum over all possible $J(\gamma)$ and \mathbf{w}^γ to get Equation 25, as desired.

The claim for products of theta functions follows by almost the same argument, except that instead of summing over broken lines γ , we sum over s -tuples of broken lines γ_i with ends (p_i, Q) , $i = 1, \dots, s$. The multiplicity at Q from Equation 18 is of course just the result of writing each contribution $\prod_{i=1}^s c_{\gamma_i} z^{n_{\gamma_i}}$ as a single monomial in the quantum torus algebra, rather than as a product of q -commuting monomials. Finally, the claim about the structure constants now follows immediately from Equation 11. \square

3.6. Scattering diagrams and tropical curves. Here we briefly describe the scattering diagram \mathfrak{D} itself (as opposed to the corresponding broken lines) directly in terms of tropical curve and disk counts.⁴ Let \mathbf{m} , \mathbf{m}_{ij} , and \mathbf{w} be as in §3.3 with $\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \emptyset}(n)$, and let L be a generic line in $\mathcal{M}_{\mathbb{R}}$ intersecting n^\perp transversely. Recall that for $s = 0$ we have defined $\mathfrak{T}_{\mathbf{m}, \emptyset, \mathbf{w}}(\mathbf{m}_{ij}, L)$ and the corresponding count $N_{\mathbf{m}, \emptyset, \mathbf{w}}^{\text{trop}}(L)$. We similarly define $\tilde{\mathfrak{T}}_{\mathbf{m}, \emptyset, \mathbf{w}}(\mathbf{m}_{ij}, L)$ and $\tilde{N}_{\mathbf{m}, \emptyset, \mathbf{w}}^{\text{trop}}(L)$ by replacing the univalent vertex Q_{out} with an unbounded edge E_{out} which we require to be contained in $L + \mathbb{R}\pi_1(n)$ (in place of the condition $h(Q_{\text{out}}) \in L$). Note that $\tilde{\mathfrak{T}}_{\mathbf{m}, \emptyset, \mathbf{w}}(\mathbf{m}_{ij}, L)$ is a set of tropical curves (as opposed to tropical disks) and thus has a Gromov-Witten theoretic interpretation.

The arguments of §3.5 are easily modified to prove:

Theorem 3.13. *Let $n \in \mathcal{N}$, $L \subset \mathcal{M}_{\mathbb{R}}$ a generic line intersecting n^\perp transversely at a point Q . Here, L generic means that $L \cap n^\perp \notin \text{Joints}(\mathfrak{D})$. Define*

$$g_Q := \prod_{\substack{(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D} \\ Q \in \mathfrak{d}}} g_{\mathfrak{d}}$$

Then for any generic choice of m_{ij} 's which are close to 0 relative to Q , we have

$$\log g_Q = \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \emptyset}(kn)} N_{\mathbf{m}, \emptyset, \mathbf{w}}^{\text{trop}}(L) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|} \hat{z}^{kn}.$$

⁴Related results include [GPS10, Theorem 2.8] (the two-dimensional classical version), [CPS, Prop. 5.14] (a classical version for more general tropical spaces than we consider), and [FS15, Corollary 4.8] (the two-dimensional quantum version).

Taking the limit as we translate L to $L - t\pi_1(n)$ for $t \rightarrow \infty$, we get that if Q is in a (possibly infinitesimal)⁵ stratum of \mathfrak{D} containing $\mathbb{R}_{\geq 0}(-\pi_1(n))$, then

$$\log g_Q = \sum_{k>0} \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \emptyset}(kn)} \tilde{N}_{\mathbf{m}, \emptyset, \mathbf{w}}^{\text{trop}}(L) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|} z^{kn}.$$

4. THE QUANTUM FROBENIUS MAP

Prior to the definition of the theta functions in [GHKK14], [FG09, §4] predicted their existence and conjectured several properties they should satisfy. Among these properties are certain symmetries under a certain (quantum) Frobenius automorphism. The purpose of this section is to prove that the classical and quantum theta functions of §2 satisfy these symmetries. We assume that our scattering diagrams are of the standard form introduced in §3.1.

The following is a more general version of [FG09, §4.1, Equation 66], where it was called the “Frobenius Conjecture.”

Theorem 4.1 (Frobenius Conjecture, classical version). *For any prime p and any $u \in \mathcal{N}$, the classical (i.e., $q = 1$) theta functions satisfy*

$$\vartheta_u^p \equiv \vartheta_{pu} \pmod{p}.$$

Proof. Consider broken lines with attached monomials az^n and az^{pn} ($a \in \mathbb{Z}$, $n \in N$) crossing a wall with associated wall-crossing automorphism ν . By [GHKK14, Theorem 1.28], we can assume the representative of the equivalence class of \mathfrak{D} was chosen so that $\nu(az^n) = az^n(1+z^f)^k$ for some $f \in N$, $k \in \mathbb{Z}_{\geq 0}$. Then $\nu(az^{pn}) = az^{pn}(1+z^f)^{pk}$. By the freshman’s dream and Fermat’s little theorem, we see that $\nu(az^{pn}) \equiv \nu(az^n)^p \pmod{p}$. It follows that the broken lines contributing to $\vartheta_{pu, Q}$ in characteristic p are the same as the broken lines contributing to $\vartheta_{u, Q}$ in characteristic p , except that the attached monomials for broken lines contributing to $\vartheta_{pu, Q}$ are the p -th powers of the corresponding attached monomials for $\vartheta_{u, Q}$. The result now follows by applying the freshman’s dream to ϑ_u^p . \square

[FG09] also predicted the following quantum version of the Frobenius Conjecture (their Conjecture 4.8.6). First we introduce some notation. Denote by $\vartheta_{u, Q}(z^n) = \sum c_n z^n \in \mathbb{k}_q((\mathcal{N}))$ the Laurent series expansion of $\vartheta_{u, Q}$ in terms of monomials z^n , $n \in N$. Then for $k \in \mathbb{Z}_{>0}$, denote $\vartheta_{u, Q}(z^{kn}) := \sum c_n z^{kn}$, the series obtained by multiplying each exponent by k . When we want to specify a certain value for q , we will write a q in the subscript, as in $\vartheta_{u, Q, q}$.

Theorem 4.2 (Frobenius Conjecture, quantum version). *Suppose q and each q^{1/d_i} are primitive k -th roots of unity for a positive odd integer k . Then for any $u \in \mathcal{N}$, we have*

$$\vartheta_{ku, Q, q}(z^n) = \vartheta_{u, Q, 1}(z^{kn})$$

The map $\vartheta_{u, Q, q}(z^n) \mapsto \vartheta_{u, Q, 1}(z^{kn})$ is what [FG09] calls the **quantum Frobenius map**.

Proof. Consider a tropical disk making a nonzero contribution to (19) for $\vartheta_{ku, Q}$. I.e., a curve Γ contributing to

$$(29) \quad \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(ku)} N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|}.$$

⁵More precisely, when we say a claim holds for Q in an infinitesimal stratum containing a certain ray, we mean that the claim holds for all strata containing that ray in \mathfrak{D}^k for any k .

The contribution is

$$(30) \quad \left(\prod_{V \in \Gamma^{[0]} \setminus Q} [\text{Mult}_\Gamma(V)]_q \right) \left(\prod_{w_{ij} \in \mathbf{w}(\Gamma)} \frac{(-1)^{w_{ij}-1}}{w_{ij}[w_{ij}/d_i]_q} \right) \frac{1}{|\text{Aut}(\mathbf{w}(\Gamma))|}.$$

Note that $\#(\Gamma^{[0]} \setminus \{Q\}) = \#\mathbf{w}(\Gamma)$. Hence, there are an equal number of $[a]_q$ -type factors in the numerator and denominator, so we can replace $[a]_q := \frac{a^q - a^{-q}}{q - q^{-1}}$ with $[a]_q := a^q - a^{-q}$ without changing (30). We use this for the rest of the proof.

The initial segment of the broken line corresponding to Γ has weight a multiple of k . We show by induction that the same is true for every edge of Γ . Let S be a maximal subset of $\Gamma \setminus Q_{\text{out}}$ such that each edge $E \in S$ has weight a multiple of k and the closure of $\Gamma \setminus S$ is connected. Suppose S is not all of $\Gamma \setminus Q_{\text{out}}$. Then S is a union of trees that each contain exactly 1 univalent vertex, with the remainder of the vertices being trivalent. The number of vertices of S is then equal to the number of unbounded edges in S . Since S contains the unbounded edge corresponding to the initial direction of the broken line, this means that Γ has more vertices of multiplicity a multiple of k than elements of $\mathbf{w}(\Gamma)$ that are a multiple of k . But for ζ a primitive k -th root of unity, $\lim_{q \rightarrow \zeta} \frac{[a]_q}{[b]_q} = 0$ if a is a multiple of k and b is not, and the limit equals a finite nonzero number (see below) if both a and b are multiples of k . Hence, the contribution of such a curve would be 0. So every edge of Γ must have been weight a multiple of k .

We now see that a tropical curve contributes to $\vartheta_{ku,Q,q}$ if and only if it is k times a tropical curve contributing to $\vartheta_{u,Q,1}$. Multiplying each edge by k takes each vertex multiplicity $[a]_q$ to $[k^2a]_q$, and each $R_{w_{ij},d_i;q} = \frac{(-1)^{w_{ij}-1}}{w_{ij}[w_{ij}/d_i]_q}$ to $R_{kw_{ij},d_i;q} = \frac{(-1)^{kw_{ij}-1}}{kw_{ij}[kw_{ij}/d_i]_q}$. As before, we can pair the trivalent vertices up with the w_{ij} 's and compute, for ζ a primitive k -th root of unity,

$$\begin{aligned} \lim_{q \rightarrow \zeta} [k^2a]_q R_{kw_{ij},d_i;q} &= \lim_{q \rightarrow \zeta} \frac{(q^{k^2a} - q^{-k^2a})(-1)^{kw_{ij}-1}}{kw_{ij}(q^{kw_{ij}/d_i} - q^{-kw_{ij}/d_i})} \\ &= \frac{(-1)^{kw_{ij}-1}}{kw_{ij}} \lim_{q \rightarrow \zeta} q^{kw_{ij}/d_i - k^2a} \frac{(q^{2k^2a} - 1)}{(q^{2kw_{ij}/d_i} - 1)}. \end{aligned}$$

Since q^{1/d_i} was also assumed to be a primitive k -th root of unity, $\lim_{q \rightarrow \zeta} q^{k/d_i} = 1$. Using this and L'Hospital's rule, the above now further simplifies to

$$\frac{(-1)^{kw_{ij}-1}}{kw_{ij}} \lim_{q \rightarrow \zeta} \frac{2k^2aq^{2k^2a-1}}{(2kw_{ij}/d_i)q^{2kw_{ij}/d_i-1}} = \frac{ad_i(-1)^{kw_{ij}-1}}{w_{ij}^2} = \frac{ad_i(-1)^{w_{ij}-1}}{w_{ij}^2},$$

where the last equality used the assumption that k is odd. This is equal to $[a]_1 R_{w_{ij},d_i;1}$, and the result follows. \square

Remark 4.3. Theorems 4.1 and 4.2 are related by [FG09, Conjecture 4.8.6] and their remarks that follow the conjecture.

5. THE CASE OF CLUSTER VARIETIES

We now introduce the cluster varieties defined in [FG09], along with their quantization from [FG09] and [BZ05]. We then describe how to apply the constructions of the previous sections to cluster varieties to get the theta functions of [GHKK14].

Definition 5.1. A **seed** is a collection of data

$$S = \{N, I, E := \{e_i\}_{i \in I}, F, \{\cdot, \cdot\}, \{d_i\}_{i \in I}\},$$

where N is a finitely generated free Abelian group, I is a finite index set, E is a basis for N indexed by I , F is a subset of I , $\{\cdot, \cdot\}$ is a skew-symmetric \mathbb{Q} -valued bilinear form, and the d_i 's are positive rational numbers called **multipliers**. We call e_i a **frozen** vector if $i \in F$.

We define another bilinear form on N by

$$B(e_i, e_j) := \epsilon_{ij} := d_j \{e_i, e_j\},$$

and we require that $\epsilon_{ij} \in \mathbb{Z}$ for all $i, j \in I$. Let $M := N^*$. Define

$$B_1 : N \rightarrow M, v \mapsto B(v, \cdot)$$

For $a \in \mathbb{R}$, define $[a]_+ := \max(a, 0)$. Given a seed S as above and a choice of $j \in I \setminus F$, we can use a **mutation** to define a new seed $\mu_j(S) := (N, I, E' = \{e'_i\}_{i \in I}, F, \{\cdot, \cdot\}, \{d_i\})$, where the (e'_i) 's are defined by

$$e'_i := \mu_j(e_i) := \begin{cases} e_i + [\epsilon_{ij}]_+ e_j & \text{if } i \neq j \\ -e_j & \text{if } i = j \end{cases}$$

Corresponding to a seed S , we can define a so-called seed \mathcal{X} -torus $\mathcal{X}_S := T_M := \text{Spec } \mathbb{k}[N]$, and a seed \mathcal{A} -torus $\mathcal{A}_S := T_N := \text{Spec } \mathbb{k}[M]$. We define **cluster monomials** $A_i := z^{e_i^*} \in \mathbb{k}[M]$ and $X_i := z^{e_i} \in \mathbb{k}[N]$, where $\{e_i^*\}_{i \in I} \subset M$ is the dual basis to E .

For any $j \in I$, we have a birational morphism $\mu_{j, \mathcal{A}} : \mathcal{A}_S \rightarrow \mathcal{A}_{\mu_j(S)}$ (called an \mathcal{A} -mutation) defined by

$$A_j \mu_{j, \mathcal{A}}^*(A'_j) = \prod_{i: \epsilon_{ji} > 0} A_i^{\epsilon_{ji}} + \prod_{i: \epsilon_{ji} < 0} A_i^{-\epsilon_{ji}}; \quad \mu_{j, \mathcal{A}}^* A'_i = A_i \text{ for } i \neq j.$$

Similarly, we have an \mathcal{X} -mutation $\mu_{j, \mathcal{X}} : \mathcal{X}_S \rightarrow \mathcal{X}_{\mu_j(S)}$ defined by

$$\mu_{j, \mathcal{X}}^*(X'_i) = X_i \left(1 + X_j^{\text{sgn}(-\epsilon_{ij})}\right)^{-\epsilon_{ij}} \text{ for } i \neq j; \quad \mu_{j, \mathcal{X}}^* X'_j = X_j^{-1}.$$

Now, the cluster \mathcal{A} -variety \mathcal{A} is defined by using compositions of \mathcal{A} -mutations to glue $\mathcal{A}_{S'}$ to \mathcal{A}_S for every seed S' which is related to S by some sequence of mutations. We similarly define the cluster \mathcal{X} -variety \mathcal{X} , with \mathcal{X} -tori and \mathcal{X} -mutations. The **cluster algebra** $\text{ord}(\mathcal{A})$ is the subalgebra of $\mathbb{k}[M]$ generated by the cluster variables A_i of every seed that we can get to by some sequence of mutations. The ring $\text{up}(\mathcal{A})$ of all global regular functions on \mathcal{A} is called the **upper cluster algebra**. The **Laurent phenomenon** says that $\text{ord}(\mathcal{A}) \subseteq \text{up}(\mathcal{A})$. We will similarly write $\text{up}(\mathcal{X})$ for the ring of global regular functions on \mathcal{X} .

5.1. Quantizing mutations. We now describe mutations in terms of more general coordinates, rather than just in terms of the cluster coordinates. As in [GHK13], for a lattice \mathcal{M} with dual \mathcal{N} and with $u \in \mathcal{M}$, $\psi \in \mathcal{N}$, and $\psi(u) = 0$, define

$$m_{\mathcal{M}, u, \psi} : T_{\mathcal{M}} \dashrightarrow T_{\mathcal{M}} \\ m_{\mathcal{M}, u, \psi}^*(z^n) = z^n (1 + z^\psi)^{-n(u)} \quad \text{for } n \in \mathcal{N}.$$

One can check that the mutations above satisfy

$$(31) \quad \begin{aligned} \mu_{j, \mathcal{X}}^* &= m_{M, B(\cdot, e_j), e_j}^* : z^n \mapsto z^n (1 + z^{e_j})^{-B(n, e_j)} \\ \mu_{j, \mathcal{A}}^* &= m_{N, e_j, B(e_j, \cdot)}^* : z^m \mapsto z^m (1 + z^{B(e_j, \cdot)})^{-m(e_j)}. \end{aligned}$$

We now describe how to quantize these mutations. First, recall the quantum torus $\mathcal{T}_{\mathcal{M}, q^{1/D}, \mathbf{W}} := \mathbb{k}_q[\mathcal{N}]$ from §1.5.2, as well as the function

$$\Psi_q(x) := \exp(-\mathrm{Li}_2(-x; q)) = \prod_{a=1}^{\infty} \frac{1}{1 + q^{2a-1}x}$$

from §2.2. Let c be a positive integer dividing D . We define a homomorphism $m_{\mathcal{M}, \psi, \mathbf{W}; q^{1/c}} : \mathcal{T}_{\mathcal{N}, q^{1/D}, \mathbf{W}} \dashrightarrow \mathcal{T}_{L, q^{1/D}, \mathbf{W}}$ by:

$$m_{L, \psi, \mathbf{W}; q^{1/c}}^*(z^n) := \Psi_{q^{1/c}}(z^\psi) z^n \Psi_{q^{1/c}}(z^\psi)^{-1} = \prod_{a=1}^{c|\mathbf{W}(n, \psi)|} \left(1 + q^{\mathrm{sgn}(\mathbf{W}(n, \psi))(2a-1)/c} z^\psi\right)^{-\mathrm{sgn}(\mathbf{W}(n, \psi))} z^n.$$

One easily sees that the $q^{1/D} \rightarrow 1$ limit of this is $m_{L, u, \psi}$ for $u := c\mathbf{W}(\cdot, \psi)$. If $\mathcal{N} = N$, $\mathcal{M} = M$, $\mathbf{W} = \{\cdot, \cdot\}$, $\psi = e_j$, and $c = d_j$, then $c\mathbf{W}(\cdot, \psi) = B(\cdot, e_j)$. This suggests the quantum \mathcal{X} -mutation

$$\mu_{j, \mathcal{X}; q} := m_{M, z^{e_j}, \{\cdot, \cdot\}; q^{1/d_j}}.$$

Indeed, this is the quantized \mathcal{X} -mutation from [FG09].

Note that $m_{\mathcal{M}, u, \psi} = m_{\mathcal{M}, -u, \psi}^{-1}$. Instead of directly quantizing $m_{N, e_j, B(e_j, \cdot)}$, we quantize $m_{N, -e_j, B(e_j, \cdot)}$ in order to match the quantization from [BZ05]. In analogy with the above situation, we want a skew-symmetric form Λ on M such that if we take $\mathcal{N} = M$, $\mathcal{M} = N$, $\mathbf{W} = \Lambda$, $\psi = B(e_j, \cdot)$, and $c = d_j$, then $c\mathbf{W}(\cdot, \psi) = -e_j$. That is, we want Λ to satisfy

$$\Lambda(B_1(e_j), \cdot) = \frac{1}{d_j} e_j \quad \text{for } j \in I \setminus F.$$

e_j here is viewed here as an element of the dual to M . Such a Λ does not always exist—One can show that its existence requires the restriction of B_1 to the span $\langle e_i \rangle_{i \in I \setminus F}$ to be injective (i.e., the “principal part of the exchange matrix” must be nondegenerate). In terms of matrices, this means that $B\Lambda = D$ for some D such that $D_{ij} = \frac{1}{d_i} \delta_{ij}$ for i or j in $I \setminus F$. When such a Λ does exist, (B, Λ) is what [BZ05] calls a **compatible pair**. We will call a cluster \mathcal{A} -variety **quantizable** if a corresponding seed admits a compatible pair. The quantum \mathcal{A} -mutation is now defined by:

$$\mu_{j, \mathcal{A}; q} := m_{N, z^{B(e_j, \cdot)}, \Lambda; q^{1/d_j}}^{-1}.$$

We note that [GHKK14] does not use the notion of a compatible pair as they work mostly in the classical limit. When we state one of their results holds for quantizable \mathcal{A} , it is actually the weaker condition of $B_1(I \setminus F)$ being contained in a strictly convex cone that is needed.

Example 5.2. Suppose $\{\cdot, \cdot\}$ is non-degenerate. Then B is also non-degenerate, and we can take $\Lambda = B^{-1}D$ for D the diagonal matrix with $D_{ii} = \frac{1}{d_i}$. This can be viewed as the form on $M_{\mathbb{Q}} = B_1(N_{\mathbb{Q}})$ induced by $\{\cdot, \cdot\}$ —that is, $\Lambda(B_1(u), B_1(v)) := \{v, u\}$ (note the order reversal of u and v). In particular, we will use that:

$$(32) \quad \Lambda(B_1(e_i), \cdot) = \frac{1}{d_i} e_i.$$

We denote by $\mathrm{up}(\mathcal{A}_q)$ and $\mathrm{up}(\mathcal{X}_q)$, or simply by \mathcal{A}_q and \mathcal{X}_q , the quantizations of $\mathrm{up}(\mathcal{A})$ and $\mathrm{up}(\mathcal{X})$ —i.e., the subrings of $\mathcal{T}_{N, q^{1/D}, \Lambda}$ and $\mathcal{T}_{M, q^{1/D}, \{\cdot, \cdot\}}$, respectively, which are closed under all sequences of quantized mutations from the base seed S . Similarly, $\mathrm{ord}(\mathcal{A}_q)$ is the subring of $\mathrm{up}(\mathcal{A}_q)$ obtained by, for each S' mutation equivalent to S , mutating the corresponding $z^{e_i^*}$'s back to the copy $\mathcal{T}_{N, q^{1/D}, \Lambda}$ corresponding to S , and taking the ring generated by all these.

5.2. Scattering diagrams for \mathcal{A}_q and \mathcal{X}_q . We now describe the initial scattering diagrams used for constructing the theta functions associated with \mathcal{A}_q and \mathcal{X}_q .

5.2.1. The initial scattering diagram for \mathcal{A}_q . When constructing quantized theta functions in \mathcal{A}_q (assuming a form Λ compatible with B exists), we take $\mathcal{N} := M$, $\mathcal{M} := N$, and $\mathbf{W} := \Lambda$. Our basis $\{f_i | i \in I\}$ for \mathcal{N} is chosen so that $f_i := B_1(e_i)$ for $i \in I \setminus F$, and we take d_i as in the seed data for $i \in I \setminus F$. There is flexibility for $i \in F$, but in the situation of Example 5.2, we can take $f_i := B_1(e_i)$ and d_i as in the seed data for all $i \in I$. Our initial scattering diagram is then:

$$(33) \quad \mathfrak{D}_{\mathcal{A}_q, \text{in}} := \left\{ (B_1(e_i)^\perp, [\Psi_{q^{1/d_i}}(z^{B_1(e_i)})]^{-1}) | i \in I \setminus F \right\}.$$

Note that crossing $B_1(e_i)^\perp$ from the $B_1(e_i) > 0$ side to the $B_1(e_i) < 0$ side corresponds to applying the mutation $\mu_{i, \mathcal{A}; q}^{-1}$. We write $\widehat{B}_{\mathcal{A}_q}$ for the algebra generated by the theta functions $\{\vartheta_p | p \in M\}$ constructed from $\mathfrak{D}_{\mathcal{A}_q} := \text{Scat}(\mathfrak{D}_{\mathcal{A}_q, \text{in}})$.

5.2.2. The initial scattering diagram for \mathcal{X}_q . When constructing quantized theta functions in \mathcal{X}_q , we take $\mathcal{N} := N$, $\mathcal{M} := M$, $\mathbf{W} := \{\cdot, \cdot\}$, and $\{d_i | i \in I\}$ as in the seed data. The basis $\{f_i | i \in I\}$ for \mathcal{N} is the one from the seed data—i.e., $f_i := e_i$. The initial scattering diagram is then:

$$\mathfrak{D}_{\mathcal{X}_q, \text{in}} := \left\{ (e_i^\perp, \Psi_{q^{1/d_i}}(z^{e_i})) | i \in I \setminus F \right\}.$$

Note that crossing e_i^\perp from the $e_i > 0$ side to the $e_i < 0$ side corresponds to applying the mutation $\mu_{i, \mathcal{X}; q}^{-1}$. We write $\widehat{B}_{\mathcal{X}_q}$ for the algebra generated by the theta functions $\{\vartheta_p | p \in N\}$ constructed from $\mathfrak{D}_{\mathcal{X}_q} := \text{Scat}(\mathfrak{D}_{\mathcal{X}_q, \text{in}})$.

5.3. The relationship between $\widehat{B}_{\mathcal{A}_q}$ and \mathcal{A}_q . Much of [GHKK14] is concerned with understanding various aspects of the relationships between $\widehat{B}_{\mathcal{A}}$ and \mathcal{A} . We briefly describe some of this now, and we note some properties that we expect to also hold for the relationships between $\widehat{B}_{\mathcal{A}_q}$ and \mathcal{A}_q . The main obstacle to proving many of these relationships is that there is not yet a proof of positivity of the coefficients of the quantum scattering functions, a property expected to hold whenever each $d_i = 1$.

5.3.1. Constructing \mathcal{A}_q from the scattering diagram. [GHKK14, §4] describes how to construct \mathcal{A} from the scattering diagram $\mathfrak{D}_{\mathcal{A}}$. Briefly, there is a nice subset of $\mathfrak{D}_{\mathcal{A}}$ called the **cluster complex** Δ^+ whose chambers σ correspond to seeds S_σ of \mathcal{A} (here, one must view the scattering diagram as being supported in $\mathcal{N}_{\mathbb{R}}$ as in Remark 2.10). The action on T_M of the path-ordered product for going from a chamber σ' to a chamber σ is precisely the composition of the corresponding sequence of mutations relating S_σ to $S_{\sigma'}$. Thus, \mathcal{A} consists of one copy of T_M for each chamber σ in the cluster complex, and these tori are glued using path-ordered products. For $\sigma, \sigma' \in \Delta^+$, $p \in M \cap \sigma$, Q in the interior of σ' , ϑ_p corresponds to a product of cluster monomials for the seed S_σ , and $\vartheta_{p, Q}$ is the expression of this product in the seed $S_{\sigma'}$. One easily sees that this directly generalizes to the quantized situation. Thus,

$$\text{up}(\mathcal{A}_q) \subseteq \widehat{B}_{\mathcal{A}_q}$$

with the quantized cluster monomials being equal to certain theta functions.

5.3.2. The formal Fock-Goncharov conjecture. In §5.4 below, we will associate a quantizable cluster \mathcal{A} -variety $\mathcal{A}^{\text{prin}}$ to any cluster variety \mathcal{A} . [GHKK14, §6] describes a degeneration $\widehat{\mathcal{A}}^{\text{prin}}$ of $\mathcal{A}^{\text{prin}}$ (morally a large complex structure limit) such that all the theta functions in $\widehat{B}_{\mathcal{A}}$ can be viewed as functions on some formal neighborhood of the special fiber of $\widehat{\mathcal{A}}^{\text{prin}}$. We expect analogous constructions to work with \mathcal{X} and with any quantizable \mathcal{A} , and also with the quantizations \mathcal{A}_q and \mathcal{X}_q .

5.3.3. The middle cluster algebra. Suppose \mathcal{A} is quantizable and fix a base seed S . Let $\sigma := \{e_i > 0 | i \in I \setminus F\}$ be the interior of the corresponding cone of the cluster complex Δ^+ , and let $Q \in \sigma$. Let $\Theta_{\mathcal{A}} \subset M$ denote the set of $p \in M$ such that $\vartheta_{p,Q}$ is a Laurent polynomial (as opposed to an infinite Laurent series). [GHKK14, §7.1] shows that in the classical situation, $\Theta_{\mathcal{A}}$ does not depend on the choice of base seed S . Furthermore, the submodule $\mathbb{k}\langle\Theta_{\mathcal{A}}\rangle \subset \widehat{B}_{\mathcal{A}}$ is closed under multiplication—that is, it is in fact a subalgebra. This subalgebra is called $\text{mid}(\mathcal{A})$ since we have the natural inclusions $\text{ord}(\mathcal{A}) \subseteq \text{mid}(\mathcal{A}) \subseteq \text{up}(\mathcal{A})$. In §5.4 we will explain how to define $\text{mid}(\mathcal{A})$ for non-quantizable \mathcal{A} . See [GHKK14, §7.2] for more details, including the definition of $\text{mid}(\mathcal{X}) \subseteq \text{up}(\mathcal{X})$.

We expect that analogs $\text{mid}(\mathcal{A}_q)$ and $\text{mid}(\mathcal{X}_q)$ of $\text{mid}(\mathcal{A})$ and $\text{mid}(\mathcal{X})$ make sense for the quantum situation, but this has not yet been worked out. Even [GHKK14]’s proof that $\Theta_{\mathcal{A}}$ does not depend on S (their Proposition 7.1) uses positivity properties of broken lines that do not hold in non-skew-symmetric quantum situation. An analog of their Theorem 7.5 would also be desirable.

5.3.4. The full Fock-Goncharov conjecture. Ideally, one would like to have $\text{mid}(\mathcal{A}) = \text{up}(\mathcal{A}) = \widehat{B}_{\mathcal{A}}$ and $\Theta_{\mathcal{A}} = M$. When this holds in the classical situation, [GHKK14] says that *the full Fock-Goncharov conjecture holds* (cf. their Definition 0.6). They look at conditions implying the full Fock-Goncharov conjecture, cf. their Proposition 0.13. We predict that any conditions implying the classical full Fock-Goncharov conjecture will also imply the quantum analog.

5.4. Principal coefficients and non-quantizable \mathcal{A} . The condition of the existence of a compatible Λ is restrictive, but [GHKK14] gets around this restriction by working with $\mathcal{A}^{\text{prin}}$, the cluster \mathcal{A} -variety with principal coefficients. They construct theta functions on this space and then restrict to the subspace $\mathcal{A} \subset \mathcal{A}^{\text{prin}}$ where we want the theta functions. We explain this approach now.

Definition 5.3. Let $S = \{N, I, E := \{e_i\}_{i \in I}, F, \{\cdot, \cdot\}, \{d_i\}_{i \in I}\}$ be a seed corresponding to \mathcal{A} . As usual, let $M := N^*$. $\mathcal{A}^{\text{prin}}$ is the cluster \mathcal{A} -variety corresponding to the seed S^{prin} defined as follows:

- $N_{S^{\text{prin}}} := N \oplus M$.
- $I_{S^{\text{prin}}}$ is the disjoint union of two copies of I . We will call them I_1 and I_2 to distinguish between them.
- $E_{S^{\text{prin}}} := \{(e_i, 0) | i \in I_1\} \cup \{(0, e_i^*) | i \in I_2\}$
- $F_{S^{\text{prin}}} := F_1 \cup I_2$, where F_1 is simply F viewed as a subset of I_1 .
- $\{(n_1, m_1), (n_2, m_2)\}_{S^{\text{prin}}} := \{n_1, n_2\} + m_2(n_1) - m_1(n_2)$.
- The d_i ’s are the same as before (viewing i as an element of I).

One checks that $\{\cdot, \cdot\}_{S^{\text{prin}}}$ is unimodular, so in particular, there exists a Λ^{prin} as in Example 5.2 compatible with B^{prin} . We can thus construct theta functions on $\mathcal{A}^{\text{prin}}$. Furthermore, there is a map $\pi : \mathcal{A}^{\text{prin}} \rightarrow T_M$ defined on seeds by the map of cocharacter lattices $\pi : N \oplus M \rightarrow M$, $(n, m) \mapsto m$. For $t \in T_M$, let $\mathcal{A}_t := \pi^{-1}(t) \subset \mathcal{A}^{\text{prin}}$. The original \mathcal{A} -space is $\mathcal{A} = \mathcal{A}_e$, where e is the identity in T_M .

We can now construct the theta functions on $\mathcal{A}^{\text{prin}}$ and apply the techniques of §2.4 with $K := (0, N) \subset M \oplus N$. Indeed, for Λ^{prin} as in Example 5.2, $f_i := B_1^{\text{prin}}((e_i, 0))$, and any $(0, n) \in K$, Equation 32 gives us

$$\Lambda^{\text{prin}}[f_i, (0, n)] = \frac{1}{d_i} \langle (e_i, 0), (0, n) \rangle = 0.$$

Thus, we do have $\mathbf{W}(\mathcal{N}^+, K) = 0$ for $\mathbf{W} := \Lambda^{\text{prin}}$. The condition $\ker(\pi_1) \subset K$ is trivial since this \mathbf{W} is nondegenerate. Choosing a section $\tilde{\varphi}$ of the projection $\pi_K : M \oplus N \rightarrow (M \oplus N)/K \cong M$ as in §2.4,

we obtain theta functions $\vartheta_p := \vartheta_{\tilde{\varphi}(p)}$ for $p \in M$. Now Equation 12 describes the multiplication of these theta functions over $\mathbb{k}[[P_K]]$. Define $\Theta_{\mathcal{A}^{\text{prin}}}$ as in §5.3.3, and take $\Theta_{\mathcal{A}}$ and $\Theta_{\mathcal{A}_t}$ to be $\pi_K(\Theta_{\mathcal{A}^{\text{prin}}})$. One can show that $\Theta_{\mathcal{A}^{\text{prin}}}$ is closed under addition by K , hence equal to $\pi_K^{-1}(\Theta_{\mathcal{A}})$, and in particular containing $\{\tilde{\varphi}(p) | p \in \Theta_{\mathcal{A}}\}$. Multiplication of the corresponding theta functions is defined over $\mathbb{k}[[P_K]]$. Since $P_K \subset (0, N)$, we can evaluate at our point $t \in T_M$ to get a theta function multiplication rule for \mathcal{A}_t . In general, these theta functions are only independent of the section $\tilde{\varphi}$ up to scaling, but for $t = e$ they are uniquely defined.

Let us now apply Theorem 3.7 to \mathcal{A} . According to the theorem, for $p_1, \dots, p_s \in \Theta_{\mathcal{A}} \subseteq M$, we have the following formula in $\mathcal{A}^{\text{prin}}$ (using classical limits of the theta functions, $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$, and $R_{\mathbf{w}}$):

$$(34) \quad \vartheta_{\tilde{\varphi}(p_1), Q} \cdots \vartheta_{\tilde{\varphi}(p_s), Q} = \sum_{p \in \Theta_{\mathcal{A}}} \sum_{n \in N} \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(\tilde{\varphi}(p) + n)} N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|} z^{\tilde{\varphi}(p)} z^n.$$

The sum is finite by the definition of $\Theta_{\mathcal{A}}$. Evaluating at $t = e$ (i.e., setting each $z^n = 1$ and each $z^{\tilde{\varphi}(p_1)} z^{\tilde{\varphi}(p_2)} = z^{\tilde{\varphi}(p_1 + p_2)}$) gives the corresponding formula in \mathcal{A} .

In Equation 34, the tropical curves are in $\overline{\mathcal{M}}_{\mathbb{R}} = N_{\mathbb{R}} \oplus M_{\mathbb{R}}$, and $m_i := \Lambda^{\text{prin}}(f_i, \cdot) = \frac{1}{d_i}(e_i, 0)$ for $i \in I_1 \setminus F_1$. Alternatively, we can use Remark 2.10 to view the scattering diagram as living in $\mathcal{N}_{\mathbb{R}} = M_{\mathbb{R}} \oplus N_{\mathbb{R}}$, and in this way we can get a modified version of Theorem 1.1. The statement of the modified version is essentially the same, except that Q and the tropical curves now live in $\mathcal{N}_{\mathbb{R}}$, \mathbf{W} honestly denotes the form \mathbf{W} (in this case Λ^{prin}) on $\mathcal{N}_{\mathbb{R}}$ instead of the induced form on $\overline{\mathcal{M}}_{\mathbb{R}}$, $m_i := f_i$ instead of $\pi_1(f_i)$, and similarly, $\overline{p}_i := \tilde{\varphi}(p_i)$ instead of $\pi_1(\tilde{\varphi}(p_i))$.

Now, we observe that for every vertex $V \neq Q_{\text{out}}$ in any tropical curve contributing to some $N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q)$ in this modified version of the theorem, at least one of the outgoing vectors u (any one not corresponding to a direction of the associated broken line) is a sum of f_i 's. Hence, $\Lambda^{\text{prin}}(u, \cdot)$ is contained in $N \oplus \{0\}$ by Equation 32, and so the N -coordinate of either other outgoing vector $v \in M \oplus N$ does not affect the multiplicity $|\Lambda^{\text{prin}}(u, v)|$ of V . Thus, we can replace each tropical curve in (34) with its projection under π_K without affecting the multiplicities or the values of the z^n -terms. Furthermore, for generic \mathbf{m}_{ij} , any two distinct tropical curves in $\mathfrak{T}_{\mathbf{m}, \mathbf{p}, \mathbf{w}}(\mathbf{m}_{ij}, Q)$ will project under π_K to distinct tropical curves in $M_{\mathbb{R}}$. Note that $\pi_K(f_i) = B_1(e_i)$, and $\pi_K(\tilde{\varphi}(p_i)) = p_i$. Thus, we obtain the following theorem describing classical theta functions in $\text{mid}(\mathcal{A})$:

Theorem 5.4. *Consider a seed S as in Definition 5.1. In Definition 3.5, take $\overline{\mathcal{M}}$ to be M , $\mathbf{m} := (m_i := B_1(e_i))_{i \in I \setminus F}$, $\mathbf{p} := (p_1, \dots, p_s)$ an s -tuple ($s \geq 1$) of vectors in $\Theta_{\mathcal{A}} \setminus \{0\} \subset M$, and \mathbf{W} satisfying $\mathbf{W}(m_i, \cdot) = \frac{1}{d_i}e_i$ (e.g., $\mathbf{W} := \Lambda^{\text{prin}}|_{M_{\mathbb{R}} \oplus \{0\}}$). For any $m \in M$, the coefficient of z^m in the product $\vartheta_{p_1, Q} \cdots \vartheta_{p_s, Q}$, ($Q \in M_{\mathbb{R}}$ generic and sufficiently far from the origin) is*

$$(35) \quad \sum_{\mathbf{w} \in \mathfrak{W}_{\mathbf{m}, \mathbf{p}}(m)} N_{\mathbf{m}, \mathbf{p}, \mathbf{w}}^{\text{trop}}(Q) \frac{R_{\mathbf{w}}}{|\text{Aut}(\mathbf{w})|}.$$

All terms here are defined using the classical $q^{1/D} \rightarrow 1$ limit. If Q is sufficiently far from the origin but close to the ray ρ_m generated by m , or if $\mathbf{W}(m, \mathcal{N}^+) = 0$ and Q is any generic point (not necessarily far from the origin), then (35) gives the structure constant $\alpha(p_1, \dots, p_s; m)$. Furthermore, we can define $\Theta_{\mathcal{A}} \setminus \{0\}$ to be the set of $p \in M \setminus \{0\}$ such that, for $\mathbf{p} := (p)$, (35) is finite for all m and nonzero for only finitely many m .

In the above theorem, if $\langle e_i, Q \rangle$ is sufficiently large relative to each m_{ij} , then Q is in the chamber of the cluster complex corresponding to S . Thus, (35) for such a Q is the expression of $\vartheta_{p_1, Q} \cdots \vartheta_{p_s, Q}$ as

a Laurent polynomial in the cluster monomials corresponding to the base seed S . In particular, since all cluster monomials are theta functions, this gives a formula in terms of tropical curve counts for expressing any product of cluster monomials in terms of the cluster monomials of a given base seed.

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